Stabilising the supersymmetric Standard Model on the $\mathbb{Z}_6'$ orientifold

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Abstract

Four stacks of intersecting supersymmetric fractional D6-branes on the $\mathbb{Z}_6'$ orientifold have previously been used to construct consistent models having the spectrum of the supersymmetric Standard Model, including a single pair of Higgs doublets, plus three right-chiral neutrino singlets. However, various moduli, Kähler moduli and complex-structure moduli, twisted and untwisted, remain unfixed. Further, some of the Yukawa couplings needed to generated quark and lepton masses are forbidden by a residual global symmetry of the model. In this paper we study the stabilisation of moduli using background fluxes, and show that the moduli may be stabilised within the Kähler cone. In principle, missing Yukawa couplings may be restored, albeit with a coupling that is suppressed by non-perturbative effects, by the use Euclidean D2-branes that are pointlike in spacetime, *i.e.* E2-instantons. However, for the models under investigation, we show that this is *not* possible.

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1 Introduction

The attraction of using intersecting D6-branes in a bottom-up approach to constructing the Standard Model is by now well known [1], and indeed models having just the spectrum of the Standard Model have been constructed [2, 3]. The four stacks of D6-branes wrap 3-cycles of an orientifold T^6/Ω, where the six extra spatial dimensions are assumed to be compactified on a 6-torus T^6 and Ω is the world-sheet parity operator; the use of an orientifold is essential to avoid the appearance of additional vector-like matter. However, non-supersymmetric intersecting-brane models lead to flavour-changing neutral-current (FCNC) processes induced by stringy instantons that can only be suppressed to levels consistent with current bounds by choosing a high string scale, of order 10^4 TeV, which in turn leads to fine tuning problems [4]. It is therefore natural, and in any case of interest in its own right, to construct intersecting-brane models of the supersymmetric Standard Model. A supersymmetric theory is not obliged to have a low string scale, so the instanton-induced FCNC processes may be reduced to rates well below the experimental bounds by choosing a sufficiently high string scale without inducing fine-tuning problems.

To construct a supersymmetric theory [5, 6, 7, 8, 9], instead of T^6, one starts with an orbifold T^6/P, where P is a point group which acts as an automorphism of the lattice defining T^6; this has the added advantage of fixing (some of) the complex structure moduli. An orientifold is then constructed as before by quotienting the orbifold with the action of the world-sheet parity operator Ω. In previous papers [10, 11, 12] we have studied orientifolds with the point group $P = \mathbb{Z}_6'$, and derived models having the spectrum of the supersymmetric Standard Model plus three right-chiral neutrinos. The 6-torus factorises into three 2-tori $T^6 = T^2_1 \times T^2_2 \times T^2_3$, with T^2_k $(k = 1, 2, 3)$ parametrised by the complex coordinate z^k. Then the generator θ of the point group $P = \mathbb{Z}_6'$ acts on z^k as

$$\theta z^k = e^{2\pi i v^k} z^k \tag{1}$$

where

$$(v^1, v^2, v^3) = \frac{1}{6}(1, 2, -3) \tag{2}$$

This requires that T^2_1 and T^2_2 are $SU(3)$ root lattices, so that θ is an automorphism, and this in turn fixes the complex structure moduli $U_{1,2}$ for $T^2_{1,2}$ to be $U_1 = U_2 = e^{i\pi/3} \equiv \alpha$. (Note that the G_2 and $SU(3)$ root lattices are the same, contrary to our previous assertions.) Since θ acts on T^2_3 as a reflection, $\theta z^3 = -z^3$, its lattice is arbitrary. The embedding $\mathcal{R}$ of Ω acts antilinearly on all z^k and we may choose the phases so that

$$\mathcal{R} z^k = \bar{z}^k \ (k = 1, 2, 3) \tag{3}$$

This too must be an automorphism of the lattice, and this requires the fundamental domain of each torus T^2_k to be in one of two orientations, denoted **A** and **B**, relative to the $\mathrm{Re}\, z^k$ axis. In the **A** orientation of T^2_1 the basis vector $e_1 = R_1$ is real, whereas in the **B** orientation $e_1 = R_1 e^{-i\pi/6}$; for both orientations the second basis vector $e_2 = \alpha e_1$. Similarly for the basis vectors e_3 and e_4 of T^2_2. For T^2_3, the basis vector $e_5 = R_5$ is real in both orientations, but the real part of the complex structure $U_3 \equiv e_6/e_5$ satisfies $\mathrm{Re}\, U_3 = 0$ in the **A** orientation, and $\mathrm{Re}\, U_3 = 1/2$ in the **B** orientation. Thus $e_6 = iR_5 \mathrm{Im}\, U_3$ in **A**, and $e_6 = R_5(1/2 + i\mathrm{Im}\, U_3)$ in **B**.

The models having the spectrum of the supersymmetric Standard Model, with which we are concerned in this paper, arise only in the **AAA** and **BAA** orientations. They include four stacks of (supersymmetric) fractional D6-branes, each wrapping the three large spatial dimensions and a 3-cycle of the general form

$$\kappa = \frac{1}{2}\left(\Pi^{\rm bulk}_\kappa + \Pi^{\rm ex}_\kappa\right) \tag{4}$$

where

$$\Pi^{\rm bulk}_\kappa = \sum_{p=1,3,4,6} A^\kappa_p \rho_p \tag{5}$$

is an untwisted point-group invariant bulk 3-cycle, and

$$\Pi^{\rm ex}_\kappa = \sum_{j=1,4,5,6} (\alpha_j \epsilon_j + \tilde{\alpha}_j \tilde{\epsilon}_j) \tag{6}$$

is an exceptional cycle. The four basis bulk 3-cycles ρ_p $(p = 1, 3, 4, 6)$ and their bulk coefficients A^κ_p are defined in [10], the latter being expressed in terms of the wrapping numbers (n^κ_k, m^κ_k) of the basis 1-cycles π_{2k-1}, π_{2k} of T^2_k $(k = 1, 2, 3)$. θ^3 acts as a $\mathbb{Z}_2$ reflection in T^2_1 and T^2_3 and therefore has sixteen fixed points at

$$f_{i,j} = \frac{1}{2}(\sigma_1 e_1 + \sigma_2 e_2) \otimes \frac{1}{2}(\sigma_5 e_5 + \sigma_6 e_6) \tag{7}$$

where $\sigma_{1,2,5,6} = 0, 1 \bmod 2$, and we use Honecker's [6, 13] notation in which $i, j = 1, 4, 5, 6$ correspond to the pairs (σ_1, σ_2) or (σ_5, σ_6)

$$1 \sim (0, 0), \quad 4 \sim (1, 0), \quad 5 \sim (0, 1), \quad 6 \sim (1, 1) \tag{8}$$

The eight exceptional 3-cycles $\epsilon_j,\ \tilde{\epsilon}_j$ $(j = 1, 4, 5, 6)$ and their coefficients $\alpha^\kappa_j,\ \tilde{\alpha}^\kappa_j$ are defined (in [10]) in terms of collapsed 2-cycles at $f_{i,j}$ times a 1-cycle in T^2_2, the coefficients being determined by the wrapping numbers (n^κ_2, m^κ_2) of the basis 1-cycles $\pi_3,\ \pi_4$ of T^2_2. Supersymmetry requires that the bulk part of the fractional brane passes through the fixed points associated with the exceptional piece. If, for example, $(n^\kappa_3, m^\kappa_3) = (1, 0) \bmod 2$, then, depending on the choice of Wilson lines, only the exceptional cycles with $\alpha_{1,4}, \tilde{\alpha}_{1,4}$ or $\alpha_{5,6}, \tilde{\alpha}_{5,6}$ non-zero are allowed; similarly, for the $(0, 1) \bmod 2$ case, only $\alpha_{1,5}, \tilde{\alpha}_{1,5}$ or $\alpha_{4,6}, \tilde{\alpha}_{4,6}$ may be non-zero, and for the $(1, 1) \bmod 2$ case, only $\alpha_{1,6}, \tilde{\alpha}_{1,6}$ or $\alpha_{4,5}, \tilde{\alpha}_{4,5}$ may be non-zero. Orientifold invariance requires that we also include D6-branes wrapping the orientifold image $\kappa' \equiv \mathcal{R}\kappa$ of each 3-cycle κ, and the action of $\mathcal{R}$ on the basis 3-cycles $\rho_p, \epsilon_j, \tilde{\epsilon}_j$ is also given in [10]. The precise form of the 3-cycles associated with the four stacks is given in [11, 12] and need not concern us for the present. D6-branes carry Ramond-Ramond (RR) charge and are coupled electrically to the 7-form RR gauge potential C_7. So too is the O6-plane, a topological defect associated with the orientifold action which has -4 units of RR charge.

The massive version of the effective supergravity describing compactified type IIA string theory in the presence of background fluxes has action [14, 15]

$$\begin{aligned} S_{IIA} &= \frac{1}{2\kappa^2_{10}} \int d^{10}x \sqrt{-g} \left(e^{-2\phi}[\mathbb{R} + 4(\partial\phi)^2 - \frac{1}{2}|H_3|^2] - [|F_2|^2 + |F_4|^2 + m_0^2] \right) \\ &- \frac{1}{2\kappa^2_{10}} \int \Big(B_2 \wedge dC_3 \wedge dC_3 + 2B_2 \wedge dC_3 \wedge F_4^{bg} + C_3 \wedge H_3^{bg} \wedge dC_3 \\ &\quad -\frac{m_0}{3} B_2 \wedge B_2 \wedge B_2 \wedge dC_3 + \frac{m_0^2}{20} B_2 \wedge B_2 \wedge B_2 \wedge B_2 \wedge B_2 \Big) \\ &\quad -\mu_6 \sum_\kappa N_\kappa \int_{\mathcal{M}_4 \times \kappa} d^7\xi \, e^{-\phi} \sqrt{-g} + \sqrt{2}\mu_6 \sum_\kappa N_\kappa \int_{\mathcal{M}_4 \times \kappa} C_7 \end{aligned} \tag{9}$$

where $2\kappa^2_{10} = (2\pi)^7 \alpha'^4$ is the 10-dimensional Newtonian gravitational constant and $\mu_6 = (2\pi)^{-6} \alpha'^{-7/2}$ is the unit of D6-brane RR charge. The sum over κ is understood to include all D6-brane stacks, their orientifold images κ', and the O6-brane π_{O6} with charge $-4\mu_6$, and N_κ is the number of D6-branes in the stack wrapping the 3-cycle κ. The field strengths associated with the Kalb-Ramond field B_2 and the RR fields $C_{1,3}$ are

$$H_3 = dB_2 + H_3^{bg} \tag{10}$$

$$F_2 = dC_1 + m_0 B_2 \tag{11}$$

$$F_4 = dC_3 + F_4^{bg} - C_1 \wedge H_3 - \frac{m_0}{2} B_2 \wedge B_2 \tag{12}$$

where H_3^{bg} and F_4^{bg} are background fluxes, and the mass m_0 is the background value of F_0. The presence of the fluxes generally deforms the original metric. The direct product of the four-dimensional Minkowski space and the compactified (Calabi-Yau) space is replaced by a warped product [16, 17] which, as we shall see, introduces a potential for (some of) the moduli. dC_1 is the Hodge dual of F_8, the field strength associated with the 7-form RR gauge field C_7. One effect of the m_0 term is that a piece of the $F_2 \wedge^* F_2$ term in (9) couples H_3^{bg} to C_7

$$F_2 \wedge^* F_2 \supset m_0 H_3^{bg} \wedge C_7 \tag{13}$$

so that this term also contributes to the C_7 tadpole equation. The requirement that there are no RR C_7 tadpoles is therefore generalised [18] to

$$\mu_6 \left(\sum_\kappa N_\kappa (\kappa + \kappa') - 4\Pi_{\mathrm{O6}} \right) + \frac{1}{4\kappa_{10}^2} \Pi_{m_0 H_3^{bg}} = 0 \tag{14}$$

where $\Pi_{m_0 H_3^{bg}}$ is the 3-cycle of which $m_0 H_3^{bg}$ is the Poincaré dual. In the models presented in [11, 12] tadpole cancellation requires that $\Pi_{m_0 H_3^{bg}}$, and hence $m_0 H_3^{bg}$, is non-zero.

In general, we must also address the question of whether the total K-theory charge [19] is zero. The presence of K-theory charge may be exhibited by the introduction of a "probe" $Sp(2) \simeq SU(2)$ brane π_{probe}. For a consistent theory we require that there are an *even* number of chiral fermions in the fundamental representation of $Sp(2)$. Thus the additional constraint [20, 8] is that

$$\sum_\kappa N_\kappa \kappa \cap \pi_{\mathrm{probe}} = 0 \bmod 2 \tag{15}$$

where the sum is over all D6-branes, but *not* including their orientifold images, and π_{probe} is any 3-cycle that is its own orientifold image

$$\pi_{\mathrm{probe}} = \pi_{\mathrm{probe}}{}' \tag{16}$$

although this may be too strong a constraint. It follows that [20, 8]

$$\pi_{\mathrm{probe}} = \frac{1}{2} \left(\Pi_{\mathrm{probe}}^{\mathrm{bulk}} + \Pi_{\mathrm{probe}}^{\mathrm{ex}} \right) \tag{17}$$

where, on the **AAA** lattice,

$$\Pi_{\mathrm{probe}}^{\mathrm{bulk}} = A_1 \rho_1 + A_4(\rho_4 + 2\rho_6) \tag{18}$$

$$\Pi_{\mathrm{probe}}^{\mathrm{ex}} = \sum_{j=1,4,5,6} \tilde{\alpha}_j (2\epsilon_j + \tilde{\epsilon}_j) \tag{19}$$

The two independent (supersymmetric) possibilities are

$$A_p = (1,0,0,0) \quad \tilde{\alpha}_j = t_0(1, t_2, 0, 0) \quad \text{or} \quad t_0(0,0,1,t_2) \tag{20}$$

$$\text{or} \quad A_p = (0,0,1,2) \quad \tilde{\alpha}_j = t_0(1, 0, t_2, 0) \quad \text{or} \quad t_0(0,1,0,t_2) \tag{21}$$

with $t_0, t_2 = \pm 1$. In our models, in particular in the model deriving from the fourth entry in Table 1 of reference [12], the contributions to the left-hand side of (15) from the stacks b and c are necessarily *even*, the former because $N_b = 2$, and the latter because it is zero. For the remaining stacks, we find that $a \cap \pi_{\mathrm{probe}} = -d \cap \pi_{\mathrm{probe}}$ for both cases (20) and (21) above. Thus the K-theory constraint (15) *is* satisfied. The same is true of the other models on the **AAA** lattice, as well as for the **BAA** cases too.

All of the models that we have considered have the attractive feature that they have the spectrum of the supersymmetric Standard Model, including a single pair of Higgs doublets, plus three right-chiral neutrino singlets. In the presence of these suitably chosen background fields m_0 and H_3^{bg} the models *are* consistent string theory vacua. Nevertheless, despite the attraction of having "realistic" spectra, they are deficient. First, there are many unfixed moduli, Kähler moduli, complex structure moduli, axions and the dilaton, all of which have unobserved massless quanta unless they are stabilised. We shall see later that the non-zero background flux H_3^{bg} required by tadpole cancellation stabilises one linear combination of the (axion) moduli. Tadpole cancellation generally ensures the absence of anomalous $U(1)$ gauge symmetries in the models; the associated gauge boson acquires a string-scale mass via the generalised Green-Schwarz mechanism, and the $U(1)$ survives only as a global symmetry. However, some of the surviving global symmetries forbid the Yukawa couplings required to generate mass terms for some of the quarks and leptons. This is the second deficiency of these models. Further, as noted previously in [11], there is a surviving unwanted $U(1)_{B-L}$ gauge symmetry, associated with baryon number B minus lepton number L. In addition, in all of the models that we constructed, the $U(1)$ stack associated with the fractional 3-cycle c has the property that $c = c'$, where c' is the orientifold image of c. This means

that the $U(1)_c$ gauge symmetry is enhanced to $SP(2) = SU(2)$, so that the models actually have as surviving gauge symmetry group $SU(3)_{\rm colour} \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. The weak hypercharge is given by $Y = \frac{1}{2}(B-L) + T_R^3$, and the matter is in the following representations $(\mathbf{n}_3, \mathbf{n}_L, \mathbf{n}_R)_{B-L}$ of $SU(3)_{\rm colour} \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$:

$$
\begin{aligned}
Q_L &= (\mathbf{3},\mathbf{2},\mathbf{1})_{\frac{1}{3}} && (22)\\
q_L^c &= (\bar{\mathbf{3}},\mathbf{1},\mathbf{2})_{-\frac{1}{3}} && (23)\\
L &= (\mathbf{1},\mathbf{2},\mathbf{1})_{-1} && (24)\\
\ell_L^c, \nu_L^c &= (\mathbf{1},\mathbf{1},\mathbf{2})_{1} && (25)\\
H_{u,d} &= (\mathbf{1},\mathbf{2},\mathbf{2})_{0} && (26)
\end{aligned}
$$

In addition, the models we have constructed cannot yield gauge coupling constant unification. A stack κ gives rise to a gauge group factor with coupling constant g_κ given [21, 22] by

$$\frac{1}{\alpha_\kappa} \equiv \frac{4\pi}{g_\kappa^2} = \frac{m_{\rm string}^3 {\rm Vol}(\kappa)}{(2\pi)^3 g_{\rm string} K_\kappa} \tag{27}$$

where ${\rm Vol}(\kappa)$ is the volume of the 3-cycle κ and $K_\kappa = 1$ for a $U(N_\kappa)$ stack. The consistency of our treatment with the supergravity approximation requires that the contribution of the bulk part of the fractional 3-cycle $\frac{1}{2}{\rm Vol}(\Pi_\kappa^{\rm bulk})$ to ${\rm Vol}(\kappa)$ is large compared to the contribution from the exceptional part $\frac{1}{2}{\rm Vol}(\Pi_\kappa^{\rm ex})$, so we need only consider the former in evaluation g_κ^2. As derived in [10], for a supersymmetric stack, the quantity

$$Z^\kappa = e_1 e_3 e_5 [A_1^\kappa - A_3^\kappa + U_3(A_4^\kappa - A_6^\kappa) + e^{i\pi/3}(A_3^\kappa + A_6^\kappa U_3)] > 0 \tag{28}$$

is real and positive. Here A_p^κ $(p = 1,3,4,6)$ are the bulk wrapping numbers, e_{2k-1} $(k = 1,2,3)$ are the basis vectors of T_k^2, and U_3 is the complex structure of T_3^2; the complex structure of $T_{1,2}^2$ is fixed by the $\mathbb{Z}_6'$ orbifold symmetry to be $U_{1,2} = e^{i\pi/3}$. Then

$$\frac{{\rm Vol}(\kappa)}{\sqrt{2{\rm Vol}(T^6/\mathbb{Z}_6')}} = \frac{Z^\kappa}{|e_1 e_3 e_5|\sqrt{|{\rm Im}\, U_3|}} \tag{29}$$

The solutions for the **AAA** lattice given in Table 1 of [12], in which $U_3 = -i/\sqrt{3}$, all have

$$Z^a = 2|e_1 e_3 e_5| \quad \text{and} \quad Z^b = |e_1 e_3 e_5| \tag{30}$$

Using equation (27) above, it follows that at the string scale $m_{\rm string}$ the coupling strengths for the $SU(3)_{\rm colour}$ and $SU(2)_L$ groups satisfy

$$\frac{\alpha_3}{\alpha_2} = \frac{1}{2} \tag{31}$$

which is clearly inconsistent with the "observed" unification $\alpha_3 = \alpha_2$ at the scale $m_X \simeq 2 \times 10^{16}$ GeV. We reach the same conclusion for the solutions on the **BAA** lattice given in Table 6 of [12], in which $U_3 = -i\sqrt{3}$. Thus, running from the string scale to the TeV scale with the three-generation supersymmetric Standard Model spectrum, none of our solutions can reproduce the measured values of the non-abelian coupling strengths of the $SU(3)_{\rm colour}$ and $SU(2)_L$ gauge groups. In fact the only supersymmetric models obtained in [10] yielding three chiral generations $3Q_L$ of quark doublets via $(a \cap b, a \cap b') = (2,1)$ or $(1,2)$, having no chiral matter in symmetric representations of the gauge groups, and not too much in antisymmetric representations, that also produce non-abelian coupling constant unification, are the two solutions on the **BAB** lattice given in Table 15 of that paper. We showed in [12] that neither model can have just the required Standard Model spectrum, but it is of interest to see what can be achieved if we relax this constraint and allow additional vector-like matter but *not* extra chiral exotics. This requires at least two $U(1)$ stacks (both of which must be d-type in the terminology of that paper). The best we can do yields two additional vector-like Higgs doublets $2(H_u + H_d)$ and four additional vector-like charged lepton singlets $4(\ell_L^c + \bar{\ell}_L^c)$, and in any case the weak hypercharge $U(1)_Y$

gauge coupling strength $\alpha_Y \neq 3\alpha_3/5$ as required by the "observed" standard-model unification. We have not pursued this any further. The one-loop gauge threshold corrections to (27) have been computed by Gmeiner and Honecker [23]. However, for the models under consideration, these are very small and the above conclusion is unaffected. Another possibility that in principle might yield a realistic model is to start with an $SU(3)_{\rm colour}$ stack a and an $SU(2)_L$ stack b satisfying $(a\cap b, a\cap b') = (3,0)$ or $(0,3)$, and to require gauge coupling constant unification $\alpha_3 = \alpha_2$. Following the work of Gmeiner and Honecker [9], we know at the outset that there are no such models that yield the standard-model spectrum *and* satisfy tadpole cancellation without the introduction of non-zero background flux H_3^{bg}. However, since we have entertained the presence of such flux, it is of interest to know how far one can get with such models. We have searched for solutions satisfying both of these criteria, but have found none.

Finally, the presence of a non-zero flux H_3^{bg} means that there may also arise a Freed-Witten anomaly [24]. In the presence of D6-branes the localised Bianchi identity associated with the stack κ imposes the constraint [25]

$$H_3^{bg} \wedge [\kappa] = 0 \tag{32}$$

where $[\kappa]$ is the 3-form that is the Poincaré dual of κ. Since H_3^{bg} is odd under the orientifold action $\mathcal{R}$, only the $\mathcal{R}$-even part of $[\kappa]$, deriving from the $\mathcal{R}$-odd part of κ, can contribute to the anomaly. We have studied this in Appendix A. Our conclusion in all cases is that there is a non-zero anomaly deriving from the $SU(3)$ stack a and also from one of the $U(1)$ stacks.

The deficiencies detailed above mean that our models can only be considered as semi-realistic. Nevertheless, it is of interest to see the extent to which the first two deficiencies can be remedied in models with a realistic spectrum. In this paper we study the fixing of moduli using background fluxes, the stability of these solutions and their consistency with the supergravity approximation in which they are derived. We also investigate the utility of non-perturbative effects, so-called E2-instantons, to stabilise axion moduli and to repair the missing Yukawa couplings.

2 Moduli stabilisation

In this and the following section we parallel the the treatment given by DeWolfe *et al.* [14] of the $\mathbb{Z}_3\times\mathbb{Z}_3$ orientifold. It has been shown by Grimm and Louis [26] that the effective four-dimensional theory deriving from type IIA supergravity compactified on a Calabi-Yau 3-fold is an $\mathcal{N}=2$ supergravity theory. The moduli space is the product of two factors, one containing the vector multiplets (which include the Kähler moduli), and the other the hypermutiplets (which include the complex structure moduli and dilaton). The metric on each space is derived from a Kähler potential, K^K and K^{cs} respectively. The orientifold projection $\mathcal{R}$ to an $\mathcal{N}=1$ supergravity reduces the size of each moduli space.

Consider first the Kähler moduli. The complexified Kähler form

$$J_c = B_2 + iJ \tag{33}$$

is *odd* under the action of $\mathcal{R}$ and can therefore be expanded in terms of the $\mathcal{R}$-odd $(1,1)$-forms. In our case, on the $\mathbb{Z}_6'$ orbifold, we have three untwisted, invariant $(1,1)$-forms w_k $(k=1,2,3)$ defined by

$$w_k \equiv dz^k \wedge d\bar{z}^k \quad (\text{no summation}) \tag{34}$$

There are also eight θ^3-twisted sector invariant harmonic $(1,1)$-forms $e_{(1,j)},\ \hat{w}_j,\ (j=1,4,5,6)$, defined as follows. Associated with each of the 16 fixed points $f_{i,j}$, defined in (7), is a localised $(1,1)$-form

$$e_{(i,j)} \equiv \omega_{k,\bar{\ell}} dz^k \wedge d\bar{z}^\ell \quad (k,\ell = 1,3) \tag{35}$$

After blowing up the fixed point using the Eguchi-Hanson EH_2 metric [27], $\omega_{k,\bar{\ell}}$ has the form

$$\omega_{k\bar{\ell}} = a(u)\delta_{k\bar{\ell}} + b(u)(z_k - Z_k)(\bar{z}_\ell - \bar{Z}_\ell) \tag{36}$$

when the fixed point $f_{i,j}$ is at $(z^1,z^3) = (Z^1,Z^3) \in T_1^2 \times T_3^2$. The functions $a(u)$ and $b(u)$ are given by

$$a(u) = u^{-1}(\lambda^4+u^2)^{-1/2}\lambda^4 \tag{37}$$

$$b(u) = a'(u) \tag{38}$$

with λ the blow-up parameter and

$$u \equiv |z^1 - Z^1|^2 + |z^3 - Z^3|^2 \tag{39}$$

Under the action of the point group generator θ these $(1,1)$ forms transform as

$$e_{(1,j)} \to e_{(1,j)} \tag{40}$$
$$e_{(4,j)} \to e_{(6,j)} \to e_{(5,j)} \to e_{(4,j)} \tag{41}$$

Thus the eight invariant θ^3-twisted $(1,1)$ forms are $e_{(1,j)}$ and

$$\hat{w}_j \equiv e_{(4,j)} + e_{(5,j)} + e_{(6,j)} \; (j = 1,4,5,6) \tag{42}$$

We denote the blow-up parameter associated with $e_{(1,j)}$ by λ_j. Point-group invariance (41) requires that $e_{(4,j)}$, $e_{(5,j)}$ and $e_{(6,j)}$ all have the same blow-up parameter, which we denote by $\hat{\lambda}_j$. All of the invariant θ^3-twisted $(1,1)$ forms given above are odd under the action of $\mathcal{R}$, so in general we may expand the complexified Kähler form as

$$J_c = \sum_{k=1,2,3} t_k i w_k + \sum_{j=1,4,5,6} (T_j i e_{(1,j)} + \hat{T}_j i \hat{w}_j) \tag{43}$$

where

$$t_k = b_k + i v_k \tag{44}$$
$$T_j = B_j + i V_j \tag{45}$$
$$\hat{T}_j = \hat{B}_j + i \hat{V}_j \tag{46}$$

$b_k, B_j, \hat{B}_j$ are associated with the Kalb-Ramond field B_2, and the Kähler moduli $v_k, V_j, \hat{V}_j$ with the Kähler form J. The Kähler potential K^K for the Kähler moduli is given by

$$\begin{aligned} K^K &= -\log \int \left(\frac{4}{3} \int_{T^6/\mathbb{Z}_6'} J \wedge J \wedge J \right) && (47) \\ &= -\log \left(\frac{32}{3} \mathrm{Vol}_6 v_1 v_2 v_3 - 16\pi^2 \mathrm{Vol}_2 \sum_j v_2 (\lambda_j^4 V_j^2 + 3\hat{\lambda}_j^4 \hat{V}_j^2) \right) && (48) \end{aligned}$$

where $\mathrm{Vol}_{6,2}$ are the coordinate volumes of T^6 and T_2^2 respectively. Thus

$$\mathrm{Vol}_6 = \prod_{k=1,2,3} \mathrm{Vol}_k \tag{49}$$

where

$$\mathrm{Vol}_k = R_{2k-1}^2 \mathrm{Im} U_k \tag{50}$$

As previously noted, the $SU(3)$ lattice used for $T_{1,2}^2$ has $U_1 = \alpha = U_2$, so that $\mathrm{Im}U_1 = \sqrt{3}/2 = \mathrm{Im}U_2$. For the models found in [11, 12], $\mathrm{Im}U_3 = -1/\sqrt{3}$ on the **AAA** lattice and $-\sqrt{3}$ on the **BAA** lattice. It is convenient to absorb the coordinate volumes into the moduli, so we make the redefinitions

$$t_k \mathrm{Vol}_k \to t_k \tag{51}$$
$$T_j \pi \lambda_j^2 \to T_j \tag{52}$$
$$\hat{T}_j \pi \hat{\lambda}_j^2 \to \hat{T}_j \tag{53}$$

and then

$$K^K = -\log \left(\frac{32}{3} v_1 v_2 v_3 - 16 \sum_j v_2 (V_j^2 + 3\hat{V}_j^2) \right) \tag{54}$$

Note that, unlike in the $\mathbb{Z}_3 \times \mathbb{Z}_3$ case discussed in [14], the twisted moduli V_j and $\hat{V}_j$ are inextricably coupled to the untwisted modulus v_2.

The complex structure moduli are obtained by expanding the holomorphic $(3,0)$-form Ω in terms of the basis 3-forms. There are four $\mathbb{Z}_6'$-invariant untwisted 3-forms, defined as in [12] by

$$\sigma_0 \equiv dz^1 \wedge dz^2 \wedge dz^3 \tag{55}$$
$$\sigma_1 \equiv dz^1 \wedge dz^2 \wedge d\bar{z}^3 \tag{56}$$
$$\sigma_2 \equiv d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^3 = \overline{\sigma}_1 \tag{57}$$
$$\sigma_3 \equiv d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 = \overline{\sigma}_0 \tag{58}$$

Hence

$$\mathcal{R}(\sigma_0 \pm \sigma_3) = \pm(\sigma_0 \pm \sigma_3) \tag{59}$$
$$\mathcal{R}(\sigma_1 \pm \sigma_2) = \pm(\sigma_1 \pm \sigma_2) \tag{60}$$

The invariant θ^3-twisted 3-forms $\omega_j, \tilde{\omega}_j$ $(j = 1, 4, 5, 6)$ are also as defined in [12] as

$$\omega_j \equiv [\alpha(e_{(4,j)} - e_{(5,j)}) + (e_{(5,j)} - e_{(6,j)})] \wedge dz_2 \tag{61}$$
$$\tilde{\omega}_j \equiv [(e_{(4,j)} - e_{(5,j)}) + \alpha(e_{(5,j)} - e_{(6,j)})] \wedge d\bar{z}_2 \tag{62}$$

Then

$$\mathcal{R}(\omega_j \mp \alpha\tilde{\omega}_j) = \pm(\omega_j \mp \alpha\tilde{\omega}_j) \text{ on } \mathbf{AAA} \tag{63}$$
$$\mathcal{R}(\tilde{\omega}_j \mp \alpha\omega_j) = \pm(\tilde{\omega}_j \mp \alpha\omega_j) \text{ on } \mathbf{BAA} \tag{64}$$

As above, it is convenient to factorise out coordinate volumes, so that the Kähler potential K^{cs} for the complex structure moduli is independent of them. Then on the **AAA** lattice we may expand the holomorphic 3-form as

$$\begin{aligned}\Omega &= \frac{1}{\sqrt{\text{Vol}_6}}[Z_0(\sigma_0 + \sigma_3) - g_0(\sigma_0 - \sigma_3) + Z_1(\sigma_1 + \sigma_2) - g_1(\sigma_1 - \sigma_2)] \\ &\quad + \sum_j \frac{1}{\pi\hat{\lambda}_j^2\sqrt{\text{Vol}_2}}[Y_j\alpha^2(\omega_j - \alpha\tilde{\omega}_j) - f_j\alpha^2(\omega_j + \alpha\tilde{\omega}_j)]\end{aligned} \tag{65}$$

On the **BAA** lattice ω_j and $\tilde{\omega}_j$ are interchanged. In both cases $Z_{0,1}$ and Y_j are associated with the $\mathcal{R}$-even forms, and $g_{0,1}, f_j$ with the $\mathcal{R}$-odd ones. It is easy to show that the complex conjugates of the twisted 3-forms are given by

$$\bar{\omega}_j = \alpha^2\tilde{\omega}_j \tag{66}$$
$$\bar{\tilde{\omega}}_j = \alpha^2\omega_j \tag{67}$$

The orientifold constraint requires that

$$\mathcal{R}\Omega = \bar{\Omega} \tag{68}$$

which gives

$$Z_{0,1},\ g_{0,1},\ Y_j,\ f_j \quad \text{are real} \tag{69}$$

The required Kähler potential is

$$\begin{aligned}K^{\text{cs}} &= -\log\left(i\int_{T^6/\mathbb{Z}_6'} \Omega \wedge \bar{\Omega}\right) \qquad (70)\\ &= -\log\left(-\frac{16}{3}(Z_0 g_0 - Z_1 g_1) + 48\sum_j Y_j f_j\right) \qquad (71)\end{aligned}$$

The $\mathcal{R}$ projection projects out half of the moduli of the $\mathcal{N}=2$ theory, including half of the universal hypermultiplet; the dilaton and one axion survive. The surviving moduli are all contained in the complexified 3-form

$$\Omega_c \equiv C_3 + 2i\mathrm{Re}(C\Omega) \tag{72}$$

where C_3 is the RR 3-form gauge potential, and C is the "compensator" that incorporates the dilaton dependence

$$C \equiv e^{-D+K^{cs}/2} \tag{73}$$

with the four-dimensional dilaton D defined by

$$e^D \equiv \sqrt{8}e^{\phi+K^K/2} \tag{74}$$

Since C_3 is even under the action of $\mathcal{R}$ we may expand it as

$$C_3 = \frac{1}{\sqrt{\mathrm{Vol}_6}}[x_0(\sigma_0+\sigma_3)+x_1(\sigma_1+\sigma_2)] + \sum_j \frac{1}{\pi\hat{\lambda}_j^2\sqrt{\mathrm{Vol}_2}} X_j\alpha^2(\omega_j - \alpha\tilde{\omega}_j) \tag{75}$$

on the **AAA** lattice; as before, in the **BAA** case we interchange ω_j and $\tilde{\omega}_j$. Expanding Ω_c as in (65), on the **AAA** lattice

$$\begin{aligned}\Omega_c &= \frac{1}{\sqrt{\mathrm{Vol}_6}}[N_0(\sigma_0+\sigma_3)-T_0(\sigma_0-\sigma_3)+N_1(\sigma_1+\sigma_2)-T_1(\sigma_1-\sigma_2)] \\ &\quad + \sum_j \frac{1}{\pi\lambda_j^2\sqrt{\mathrm{Vol}_2}}[M_j\alpha^2(\omega_j-\alpha\tilde{\omega}_j) - S_j\alpha^2(\omega_j+\alpha\tilde{\omega}_j)]\end{aligned} \tag{76}$$

with the usual interchange for the **BAA** case. Then the surviving moduli are the expansion of Ω_c in H^3_+, *i.e.* the $\mathcal{R}$-even states with moduli

$$N_k = x_k + 2iCZ_k \quad (k=0,1) \tag{77}$$

$$M_j = X_j + 2iCY_j \quad (j=1,4,5,6) \tag{78}$$

in both cases.

The potential V arising after dimensionally reducing the massive type IIA supergravity is

$$V = e^K\left(\sum_{i,j=\{t_k,T_j,\hat{T}_j,N_k,M_j\}} K^{ij}D_iW\overline{D_jW} - 3|W|^2\right) + m_0e^{K^Q}\mathrm{Im}\,W^Q \tag{79}$$

where the Kähler potential $K = K^K + K^Q$ with

$$K^Q = -2\log\left(2\int \mathrm{Re}(C\Omega)\wedge{}^*\mathrm{Re}(C\Omega)\right) \tag{80}$$

It follows from (65) that on the **AAA** lattice

$$\mathrm{Re}(C\Omega) = \frac{1}{\sqrt{\mathrm{Vol}_6}}[CZ_0(\sigma_0+\sigma_3)+CZ_1(\sigma_1+\sigma_2)] + \sum_j \frac{1}{\pi\hat{\lambda}_j^2\sqrt{\mathrm{Vol}_2}}\alpha^2(\omega_j-\alpha\tilde{\omega}_j) \tag{81}$$

Also, since Ω is the holomorphic $(3,0)$-form, ${}^*\Omega = -i\Omega$ and

$$^*\mathrm{Re}(C\Omega) = \mathrm{Re}^*(C\Omega) = \frac{1}{\sqrt{\mathrm{Vol}_6}}[iCg_0(\sigma_0-\sigma_3)+iCg_1(\sigma_1+\sigma_2)] + i\sum_j \frac{1}{\pi\hat{\lambda}_j^2\sqrt{\mathrm{Vol}_2}} g_j\alpha^2(\omega_j+\alpha\tilde{\omega}_j) \tag{82}$$

so that

$$e^{-K^Q/2} = C^2e^{-K^{cs}} = e^{-2D} \tag{83}$$

where K^{cs} is given in (71), and the last equality follows from the definition (73). The same result follows on the **BAA** lattice. Like the Kähler moduli t_k, the complex structure moduli $N_{0,1}, M_j$ enter the Kähler potential only via their imaginary parts. The superpotential [28, 29, 30] is $W = W^Q + W^K$ where

$$W^Q(N_k, M_j) \equiv \int \Omega_c \wedge H_3^{bg} \tag{84}$$

$$W^K(t_k, T_j, \hat{T}_j) \equiv e_0 + \int J_c \wedge F_4^{bg} - \frac{1}{2}\int J_c \wedge J_c \wedge F_2^{bg} - \frac{m_0}{6}\int J_c \wedge J_c \wedge J_c \tag{85}$$

and

$$e_0 \equiv \int F_6^{bg} \tag{86}$$

We note that W^Q depends only on the NS-NS flux H_3^{bg} and W^K only on the RR fluxes F_n^{bg} $(n = 0, 2, 4, 6)$. H_3 is odd under the action of $\mathcal{R}$, so that, analogously to (75), we may expand its background value as

$$H_3^{bg} = \frac{i}{\sqrt{\mathrm{Vol}_6}}[p_0(\sigma_0 - \sigma_3) + p_1(\sigma_1 - \sigma_2)] + \sum_j \frac{i}{\pi\hat{\lambda}_j^2\sqrt{\mathrm{Vol}_2}} P_j \alpha^2(\omega_j + \alpha\tilde{\omega}_j) \tag{87}$$

on the **AAA** lattice, with $\omega \leftrightarrow \tilde{\omega}$ on **BAA**. As shown in [12], flux quantisation requires that the coefficients are quantised. On the **AAA** lattice

$$(p_0, p_1) = -\frac{\pi^2\alpha'\sqrt{\mathrm{Vol}_6}}{3\sqrt{3}R_1R_3R_5}(n_3 + 3n_6, n_3 - 3n_6) \text{ with } n_3, n_6 \in \mathbb{Z} \tag{88}$$

$$P_j = -\frac{2\pi^2\alpha'\sqrt{\mathrm{Vol}_2}}{3R_3}\hat{n}_j \text{ with } \hat{n}_j \in \mathbb{Z} \tag{89}$$

where $n_{3,6}$ and $\hat{n}_j$ respectively are associated with the flux of H_3^{bg} through the 3-cycles $\rho_{3,6}$ and ϵ_j; note that $p_{0,1}, P_j$ are *independent* of the coordinate scales $R_{1,3,5}$. For the solution discussed in §5.1 of reference [12], relating to the fourth solution in Table 1 of that paper, the exceptional part of the tadpole cancellation condition (14) requires that $|n_0\hat{n}_j| = 12$ for $j = 4, 6$. Thus $|n_0| = 1, 2, 3, 4, 6, 12$. For $j = 1$, we get $|n_0\hat{n}_1| = 12|1 - t_1^c| = 0, 24$, which is always consistent with these values of n_0. Cancellation of the untwisted part proportional to $\rho_4 + 2\rho_6$ requires that the corresponding values of n_3 satisfy $|n_3| = 1296, 648, 432, 324, 216, 108$, and of n_6 satisfy $|n_6| = (1 + t_1^c)(144, 72, 48, 36, 24, 12)$. ($t_1^c = \pm 1$ is one of the Wilson lines associated with the stack c.)

Alternatively, on the **BAA** lattice

$$(p_0, p_1) = \frac{\pi^2\alpha'\sqrt{\mathrm{Vol}_6}}{9R_1R_3R_5}(n_4 + 3n_1, -n_4 + 3n_1) \text{ with } n_1, n_4 \in \mathbb{Z} \tag{90}$$

$$P_j = \frac{2\pi^2\alpha' R_3}{\sqrt{3\mathrm{Vol}_2}}\tilde{n}_j \text{ with } \tilde{n}_j \in \mathbb{Z} \tag{91}$$

where $n_{1,4}$ and $\tilde{n}_j$ respectively are associated with the flux of H_3^{bg} through the 3-cycles $\rho_{1,4}$ and $\tilde{\epsilon}_j$. In this case tadpole cancellation of the exceptional parts requires that $|n_0\tilde{n}_j| = 12$, so that $|n_0| = 1, 2, 3, 4, 6, 12$. Then, $n_4 = 0$ and the corresponding values of n_1 satisfy $|n_1| = 432, 216, 144, 108, 72, 36$.

The form (87) for H_3^{bg} gives

$$W^Q(N_k, M_j) = -\frac{8}{3}(N_0p_0 - N_1p_1) + 24\sum_j M_jP_j \tag{92}$$

The background fluxes F_2^{bg} and F_4^{bg} that appear in W^K have similar expansions. Since F_2 is odd under the action of $\mathcal{R}$ and F_4 even

$$F_2^{bg} = \sum_{k=1,2,3} \frac{1}{\mathrm{Vol}_k} f_k i w_k + \sum_{j=1,4,5,6}\left(\frac{F_j}{\pi\lambda_j^2} i e_{(1,j)} + \frac{\hat{F}_j}{\pi\hat{\lambda}_j^2} i\hat{w}_j\right) \tag{93}$$

$$
\begin{aligned}
F_4^{bg} &= \frac{1}{\mathrm{Vol}_6}\sum_{k=1,2,3}\mathrm{Vol}_k e_k \tilde{w}_k + \sum_{j=1,4,5,6}\left(\frac{E_j}{\mathrm{Vol}_2\pi\lambda_j^2}w_2\wedge e_{(1,j)} + \frac{\hat{E}_j}{\mathrm{Vol}_2\pi\hat{\lambda}_j^2}w_2\wedge\hat{w}_j\right)\\
&+ \sum_{j=1,4,5,6}\left(\frac{G_j}{\pi^2\lambda_j^4}e_{(1,j)}\wedge e_{(1,j)} + \frac{\hat{G}_j}{\pi^2\hat{\lambda}_j^4}\hat{w}_j\wedge\hat{w}_j\right) \qquad (94)
\end{aligned}
$$

where

$$
\tilde{w}_k = dz^i\wedge d\bar{z}^i\wedge dz^j\wedge d\bar{z}^j \quad \text{where} \quad (i,j,k) = \text{cyclic }(1,2,3) \qquad (95)
$$

The constant term e_0 in W^K, defined in (86), arises from the Hodge dual F_6^{bg} of F_4 polarised in the *non*-compact directions. All of these fluxes, including F_6^{bg}, are quantised, the general constraint being that for any closed $(p+2)$-cycle Σ_{p+2}

$$
\mu_p \int_{\Sigma_{p+2}} F_{p+2} = 2\pi n \quad \text{with} \quad n\in\mathbb{Z} \qquad (96)
$$

with $\mu_p = (2\pi)^{-p}\alpha'^{-(p+1)/2}$ the electric charge of a Dp-brane. For the present, we set $F_2^{bg} = 0$, and then

$$
\begin{aligned}
W^K(t_k, T_j, \hat{T}_j) &= e_0 - \frac{4}{3}\sum_{k=1}^{3} t_k e_k + 4\sum_j (T_j E_j + 3\hat{T}_j\hat{E}_j) + 4t_2\sum_j (G_j + 3\hat{G}_j)\\
&\quad -m_0\left(\frac{4}{3}t_1t_2t_3 - 2\sum_j t_2(T_j^2 + 3\hat{T}_j^2)\right) \qquad (97)
\end{aligned}
$$

The advantage of this formalism is that we may immediately identify supersymmetric vacua by their vanishing F-terms:

$$
F_i = D_iW \equiv \partial_i W + W\partial_i K = 0 \qquad (98)
$$

for every chiral superfield i. For the complex-structure moduli, taking $i = N_k, M_j$, we get

$$
p_k + 2ie^{2D}W(Cg_k) = 0 \quad (k=0,1) \qquad (99)
$$

$$
P_j + 2ie^{2D}W(Cf_j) = 0 \quad (j=1,4,5,6) \qquad (100)
$$

As in [14], the imaginary parts of these equations are degenerate. Using (84) and (77) ... (78), they give the single constraint

$$
\mathrm{Re}\, W = 0 \qquad (101)
$$

which fixes only *one* linear combination of the axions x_0, x_1 and X_j

$$
\frac{8}{3}(x_0p_0 - x_1p_1) - 24\sum_j X_jP_j = \mathrm{Re}W^K \qquad (102)
$$

This degeneracy derives from the fact that the coeffcients p_k, P_j that determine H_3^{bg} are *real*, and therefore have insufficient degrees of freedom to stabilise both the complex structure moduli and their axionic partners. As we shall discuss later, in §5, E2-instantons can lift the remaining degeneracy. The real parts give

$$
e^{-K^{\mathrm{cs}}/2}\frac{p_k}{g_k} = e^{-K^{\mathrm{cs}}/2}\frac{P_j}{f_j} = Q_0 \qquad (103)
$$

where

$$
Q_0 \equiv \mathrm{Im}W e^D \qquad (104)
$$

Then (103) determines the moduli g_k, f_j up to an overall scale fixed by Q_0. Finally, using (83), (104) gives

$$
e^{-\phi} = \sqrt{8}e^{K^K/2}\frac{\mathrm{Im}W}{Q_0} \qquad (105)
$$

which fixes the dilaton once the other moduli are all fixed [14]. It follows from (99), (100), (71), (83) and (92) that

$$\mathrm{Im}W^Q + 2iW = 0 \tag{106}$$

Thus, using (101), when the complex structure moduli satisfy their field equations

$$2\mathrm{Im}W^K + \mathrm{Im}W^Q = 0 \tag{107}$$

and the vacuum value of the superpotential is determined entirely by the Kähler moduli

$$W = -i\mathrm{Im}W^K(t_k, T_j, \hat{T}_j) \tag{108}$$

Vanishing F-terms for the Kähler moduli in (98) give

$$\begin{aligned} e_k + m_0\frac{t_1t_2t_3}{t_k} - 4iWe^{K^K}\frac{v_1v_2v_3}{v_k} &= \frac{3}{2}\delta_{k2}\sum_j\Big[m_0(T_j^2+3\hat{T}_j^2) \\ &+ 2(G_j+3\hat{G}_j) - 4iWe^{K^K}(V_j^2+3\hat{V}_j^2)\Big] \end{aligned} \tag{109}$$

$$E_j + m_0t_2T_j = 4iWe^{K^K}v_2V_j \tag{110}$$

$$\hat{E}_j + m_0t_2\hat{T}_j = 4iWe^{K^K}v_2\hat{V}_j \tag{111}$$

Using (101), the imaginary parts of these equations require that

$$\mathrm{Im}(\partial_i W^K) = 0 \quad \text{for} \quad i = t_k, T_j, \hat{T}_j \tag{112}$$

The simplest solution of these is

$$b_k = 0 = B_j = \hat{B}_j \tag{113}$$

and then the above equations reduce to

$$e_k = \frac{3}{2}\delta_{k2}\sum_j[2(G_j+3\hat{G}_j) - X(V_j^2+3\hat{V}_j^2)] + X\frac{v_1v_2v_3}{v_k} \tag{114}$$

$$E_j = Xv_2V_j \tag{115}$$

$$\hat{E}_j = Xv_2\hat{V}_j \tag{116}$$

where

$$X \equiv m_0 + 4iWe^{K^K} = m_0 + 4\mathrm{Im}W^Ke^{K^K} \tag{117}$$

using (108). They couple the untwisted volume modulus v_2 to the twisted volume moduli $V_j, \hat{V}_j$. Solving for all moduli in terms of v_2 and X gives

$$v_1 = \frac{e_3}{Xv_2} \tag{118}$$

$$v_3 = \frac{e_1}{Xv_2} \tag{119}$$

$$V_j = \frac{E_j}{Xv_2} \tag{120}$$

$$\hat{V}_j = \frac{\hat{E}_j}{Xv_2} \tag{121}$$

Substituting these into the v_2 equation gives

$$\tilde{e}_2Xv_2^2 = e_1e_3 - \frac{3}{2}\sum_j(E_j^2+3\hat{E}_j^2) \equiv F(e_k, E_j, \hat{E}_j) \tag{122}$$

where

$$\tilde{e}_2 \equiv e_2 - 3\sum_j(G_j+3\hat{G}_j) \tag{123}$$

Then (54) gives

$$e^{K^K} = \frac{3X}{32\tilde{e}_2 v_2} \tag{124}$$

and the definition (117) yields

$$\frac{3\text{Im}W^K}{8\tilde{e}_2 v_2} = 1 - \frac{m_0}{X} \tag{125}$$

Substituting (118) ...(121) and (113) into (97), it then follows that when the Kähler moduli satisfy their field equations,

$$X = \frac{3}{5} m_0 \tag{126}$$

so that

$$|v_2| = \sqrt{\frac{5F(e_k, E_j, \hat{E}_j)}{3\tilde{e}_2 m_0}} \tag{127}$$

Thus the background fluxes $e_k, E_j, \hat{E}_j, G_j, \hat{G}_j$ and m_0 fix v_2 and X, and hence, via equations (118) ...(121), the remaining Kähler moduli.

The effective supergravity theory is a justifiable approximation [14] so long as the volumes $v_k, V_j, \hat{V}_j$ are large enough that the $\text{O}(\alpha')$ corrections are negligible and the string coupling g_s is small enough to neglect corrections. Further, to remain within the Kähler cone we require that the untwisted volumes are large compared with the blow-up volumes, *i.e.* $v_k \gg V_j, \hat{V}_j \gg 1$. Since (the non-zero value of) m_0 is fixed by the RR tadpole cancellation condition (14), and we have set $F_2^{bg} = 0$, the question then is whether there are choices of the background 4-form flux F_4^{bg} for which these constraints are obeyed.

It follows from equations (118) ... (121) that $v_{1,3}/V_j = e_{1,3}/E_j$, so that the Kähler cone constraints require that $e_1, e_3 \gg E_j$, and similarly for $\hat{E}_j$. Hence $F \sim e_1 e_3$. Then the constraints $v_k \gg 1$ require that $e_1 e_3 \gg \tilde{e}_2 m_0$, $e_1 \tilde{e}_2 \gg e_3 m_0$ and $e_3 \tilde{e}_2 \gg e_1 m_0$, and these imply that $e_1, \tilde{e}_2, e_3 \gg m_0$. For the blow-up volumes, similarly, the constraints $v_k \gg V_j, \hat{V}_j \gg 1$ require that $e_1, e_3, e_1 e_3/\tilde{e}_2 \gg E_j, \hat{E}_j \gg \sqrt{e_1 e_3/\tilde{e}_2 m_0}$. All of these are easily arranged.

3 Non-supersymmetric vacua

In general, besides the supersymmetric vacua identified in the previous section, we expect there to be additional vacua that are non-supersymmetric. To identify these we should find the effective potential in the four-dimensional Einstein frame, in which the four-dimensional Einstein-Hilbert action has the standard normalisation. However, the axion fields x_k and X_j, defined in (75), enter the ten-dimensional action (9) only via the $C_3 \wedge H_3^{bg} \wedge dC_3$ term in the Chern-Simons piece. This term is only non-zero if dC_3 is "polarised" in the four-dimensional spacetime directions, *i.e.* $dC_3 = f d^4x \equiv \mathcal{F}_0$; it has no physical degrees of freedom and can be treated as a Lagrange multiplier. The part of the action involving $\mathcal{F}_0$ has the form

$$S = -\frac{1}{2\kappa_{10}^2} \int (\mathcal{F}_0 \wedge {}^*\mathcal{F}_0 + 2\mathcal{F}_0 \wedge X) \tag{128}$$

where

$$X = F_6^{bg} + B_2 \wedge F_4^{bg} + C_3 \wedge H_3^{bg} - \frac{m_0}{6} B_2 \wedge B_2 \wedge B_2 \tag{129}$$

The equation of motion for $\mathcal{F}_0$ gives

$${}^*\mathcal{F}_0 + X = 0 \tag{130}$$

Then subsituting back gives

$$S = -\frac{1}{2\kappa_{10}^2} \int X \wedge {}^*X \tag{131}$$

which is stationary when $X = 0$. The equation that stabilises the axion follows from

$$\int X = 0 = \int \left(F_6^{bg} + B_2 \wedge F_4^{bg} + C_3 \wedge H_3^{bg} - \frac{m_0}{6} B_2 \wedge B_2 \wedge B_2 \right) \tag{132}$$

Using (43), (75), (87) and (94) this gives

$$\begin{aligned}\frac{8}{3}(x_0p_0 - x_1p_1) - 24\sum_j X_jP_j &= e_0 - \frac{4}{3}\sum_j b_je_j + 4\sum_j(B_jE_j + 3\hat{B}_j\hat{E}_j) + 4b_2\sum_j(G_j + 3\hat{G}_j) \\ &- \frac{4m_0}{3}b_1b_2b_3 + 2m_0b_2\sum_j(B_j^2 + 3\hat{B}_j^2) \end{aligned} \tag{133}$$

This fixes the *same* linear combination of the axions x_0, x_1 and X_j as in (102), and indeed, using (97), the value agrees with that found in the supersymmetric treatment when the Kalb-Ramond fields b_k, B_j and $\hat{B}_j$ have the values given in (113).

The remaining moduli are stabilised by minimising the effective potential V in the Einstein frame with metric $g^E_{\mu\nu}$. We pass to this frame by redefining the four-dimensional metric

$$g_{\mu\nu} = \frac{e^{2\phi}}{\mathrm{Vol}(\mathcal{M})}g^E_{\mu\nu} \tag{134}$$

where $\mathrm{Vol}(\mathcal{M})$ is the volume of the compact space $\mathcal{M} = T^6/\mathbb{Z}_6'$

$$\mathrm{Vol}(\mathcal{M}) \equiv \int_{T^6/\mathbb{Z}_6'} d^6y\sqrt{g_6} \tag{135}$$

with g_6 the determinant of the 6-dimensional metric. Invariance of the 6-dimensional Kähler metric under the action of the point group and the orientifold projection $\mathcal{R}$ requires that

$$ds^2 = \gamma_1 dz^1 d\bar{z}^1 + \gamma_2 dz^2 d\bar{z}^2 + \gamma_3 dz^3 d\bar{z}^3 \tag{136}$$

where the γ_i $(i = 1, 2, 3)$ are real and positive. In the θ^3-twisted sector there are 16 $\mathbb{Z}_2$ fixed points $f_{i,j} \in T^2_1 \times T^2_3$ with $i, j = 1, 4, 5, 6$, defined in (7) and (8). These fixed points are blown up using the Eguchi-Hanson EH_2 metric

$$ds^2 = g_{k,\bar{\ell}}dz^k d\bar{z}^\ell \tag{137}$$

where $k, \ell = 1, 3$ and

$$g_{k,\bar{\ell}} = \Gamma[A(u)\delta_{k\bar{\ell}} + B(u)(z_k - Z_k)(\bar{z}_\ell - \bar{Z}_\ell)] \tag{138}$$

when $f_{i,j}$ is at $(z^1, z^3) = (Z^1, Z^3) \in T^2_1 \times T^2_3$. The functions $A(u)$ and $B(u)$ are given by

$$A(u) = u^{-1}(\lambda^4 + u^2)^{1/2}\lambda^4 \tag{139}$$

$$B(u) = A'(u) \tag{140}$$

with λ the blow-up parameter and u as defined in (39). In general, both the twisted modulus Γ and the blow-up parameter λ depend on the fixed point $f_{i,j}$ with which they are associated. However, the transformation property (41) of the twisted 2-forms, or rather the analogous property of the twisted 2-cycles, shows that $\hat{\Gamma}_j$ and $\hat{\lambda}_j$, associated with $f_{4,j}, f_{5,j}$ and $f_{6,j}$, are independent of the T^2_1 fixed point $i = 4, 5, 6$; the corresponding parameters for $f_{1,j}$ are denoted by Γ_j and λ_j. In the untwisted sector there are then three real moduli and

$$\mathrm{Vol}(\mathcal{M}) = \frac{1}{6}\prod_{k=1,2,3}\mathrm{Vol}(T^2_k) = \frac{1}{6}\gamma_1\gamma_2\gamma_3\mathrm{Vol}_6 \tag{141}$$

where Vol_6 is defined in (49) and (50). The (4-dimensional) volume of the blow-up is

$$\mathrm{Vol}(f_{i,j}) = \Gamma^2\frac{1}{4}\pi^2\lambda^4 \tag{142}$$

taking $0 \leq u \lesssim \lambda^2$. The local analysis that we carry out here is valid provided that the volume of the blow-up modes is small compared with the untwisted volume $\mathrm{Vol}(T^2_1)\mathrm{Vol}(T^2_3)$ of the 4-torus containing them, *i.e.* provided that $\Gamma^2\pi^2\lambda^4 \ll \mathrm{Vol}(T^2_1)\mathrm{Vol}(T^2_3)$. Blowing up $f_{i,j}$ in this manner removes a volume

$\mathrm{Vol}(f_{i,j})$ from the untwisted volume $\mathrm{Vol}(T_1^2)\mathrm{Vol}(T_3^2)$. With $g^E_{\mu\nu}$ as given in (134), the effective potential V is defined by

$$S = \frac{1}{\kappa_{10}^2}\int d^4x\sqrt{-\det(g^E)}(-V) \tag{143}$$

Taking $F_2^{bg} = 0$, as in (97), there are four contributions to V

$$V = V_H + V_F + V_{m_0} + V_{BI} \tag{144}$$

deriving respectively from the $|H_3|^2$, $|F_4|^2$, m_0^2 and the Born-Infeld terms in (9). With H_3^{bg} given by (87), we find

$$V_H = h\frac{e^{2\phi}}{\mathrm{Vol}^2(\mathcal{M})} \tag{145}$$

where

$$h = \frac{2}{3}(p_0^2 + p_1^2) + 6\sum_j P_j^2 \tag{146}$$

on both lattices. As noted previously, h is fixed by the integers given in equations (88) ... (91), independently of the coordinate scales $R_{1,3,5}$. Similarly, with F_4^{bg} given by (94), we find

$$\begin{aligned} V_F \;\; = \;\; & \frac{e^{4\phi}}{\mathrm{Vol}^3(\mathcal{M})}\left(\frac{2}{9}\sum_{k=1,2,3} e_k^2\mathrm{Vol}(T_k^2)^2 + 16\frac{\mathrm{Vol}(\mathcal{M})}{\mathrm{Vol}(T_2^2)}\sum_j(E_j^2 + 3\hat{E}_j^2) + \right. \\ & \left. + \frac{1}{6}\mathrm{Vol}(\mathcal{M})\mathrm{Vol}(T_2^2)\sum_j\left[\frac{G_j^2}{\mathrm{Vol}(f_{(1,j)})} + \frac{3\hat{G}_j^2}{\mathrm{Vol}(f_{(4,j)})}\right]\right) \end{aligned} \tag{147}$$

where

$$\mathrm{Vol}(T_k^2) = \gamma_k \mathrm{Vol}_k \quad \text{for} \quad k = 1, 2, 3 \tag{148}$$

with Vol_k defined in (50). Likewise

$$V_{m_0} = \frac{m_0^2 e^{4\phi}}{2\mathrm{Vol}(\mathcal{M})} = \mu\frac{m_0^2 e^{4\phi}}{\mathrm{Vol}(\mathcal{M})} \tag{149}$$

with $\mu = 1/2$.

As in [14], the only terms relevant to the stabilisation of the twisted moduli are V_F and V_{m_0}, since the former dominates as $\mathrm{Vol}(f_{i,j}) \to 0$ and the latter as $\mathrm{Vol}(\mathcal{M}) \to \infty$. In equation (149) we may write

$$\mathrm{Vol}(\mathcal{M}) = \mathrm{Vol}_0(\mathcal{M}) - \frac{1}{6}\mathrm{Vol}(T_2^2)\sum_j[\mathrm{Vol}(f_{1,j}) + 3\mathrm{Vol}(f_{4,j})] \tag{150}$$

where $\mathrm{Vol}_0(\mathcal{M}) = \mathrm{Vol}(T_1^2)\mathrm{Vol}(T_2^2)\mathrm{Vol}(T_3^2)/6$ is the volume with no blow up. Then, minimising the potential gives

$$\mathrm{Vol}(f_{1,j}) = \frac{|G_j|}{\sqrt{3}|m_0|} \tag{151}$$

$$\mathrm{Vol}(f_{4,j}) = \frac{|\hat{G}_j|}{\sqrt{3}|m_0|} \tag{152}$$

and we are justified in using this local treatment provided that the F_4^{bg} fluxes are chosen so that

$$|G_j, \hat{G}_j| \ll \sqrt{3}|m_0|\mathrm{Vol}(T_1^2)\mathrm{Vol}(T_3^2) \tag{153}$$

With these values for the blow-up volume

$$V_F = V_{F1} + V_{F2} \tag{154}$$

where

$$V_{F1} = \frac{e^{4\phi}}{\mathrm{Vol}^3(\mathcal{M})}\left(\frac{2}{9}\sum_{k=1,2,3} e_k^2 \mathrm{Vol}(T_k^2)^2 + 16\frac{\mathrm{Vol}(\mathcal{M})}{\mathrm{Vol}(T_2^2)}\sum_j (E_j^2 + 3\hat{E}_j^2)\right) \tag{155}$$

$$V_{F2} = \frac{\sqrt{3}e^{4\phi}|m_0|\mathrm{Vol}(T_2^2)}{6\mathrm{Vol}^2(\mathcal{M})}\sum_j (|G_j| + 3|\hat{G}_j|) \tag{156}$$

The Born-Infeld term gives

$$V_{BI} = \mu_6\kappa_{10}^2 \frac{e^{3\phi}}{\mathrm{Vol}^2(\mathcal{M})}\sum_\kappa N_\kappa \int_\kappa d^3\xi\sqrt{\det(g_3)} \tag{157}$$

and using the (bulk part of the) tadpole cancellation condition given in (14), we can rewrite this as

$$V_{BI} = -\frac{1}{4}\frac{e^{3\phi}}{\mathrm{Vol}^2(\mathcal{M})}\int_{\Pi_{m_0 H_3^{bg}}} d^3\xi\sqrt{\det(g_3)} \tag{158}$$

where $\Pi_{m_0 H_3^{bg}}$ is the 3-cycle of which the field $m_0 H_3^{bg}$ is the Poincaré dual; H_3^{bg} is given in (87). For the two cases of interest, as shown in [12],

$$\Pi_{m_0 H_3^{bg}} = \frac{\sqrt{\mathrm{Vol}_6}}{9R_1R_3R_5} m_0\left[(p_1-p_0)\rho_1 - (p_0+p_1)(\rho_4+2\rho_6)\right] - \frac{2i\sqrt{\mathrm{Vol}_2}}{(1-2\alpha)R_3}\sum_j P_j(2\epsilon_j + \tilde{\epsilon}_j) \tag{159}$$

$$= -\frac{\sqrt{\mathrm{Vol}_6}}{9\sqrt{3}R_1R_3R_5} m_0\left[(p_0+p_1)\rho_6 + (p_1-p_0)(\rho_3+2\rho_1)\right] - \frac{2i\sqrt{\mathrm{Vol}_2}}{(1-2\alpha)R_3}\sum_j P_j(\epsilon_j + 2\tilde{\epsilon}_j) \tag{160}$$

for **AAA** and **BAA** respectively. To calculate the integral in (158), we use the result [31] quoted in [14], since $\Pi_{m_0 H_3^{bg}}$ is a special Lagrangian 3-cycle. The holomorphic 3-form Ω, defined in (65), is normalised by demanding that

$$i\int_{\mathcal{M}} \Omega\wedge\bar{\Omega} = 1 = \frac{16}{3}(Z_1 g_1 - Z_0 g_0) + 48\sum_j Y_j f_j \tag{161}$$

$$\equiv \frac{32}{3}\mathcal{G}(Z_0, Z_1, Y_j) \tag{162}$$

Then, according to the calibration formula

$$\int_{\Pi_{m_0 H_3^{bg}}} d^3\xi\sqrt{\det(g_3)} = \sqrt{2\mathrm{Vol}(\mathcal{M})}\int_{\Pi_{m_0 H_3^{bg}}} (\Omega + \bar{\Omega}) \tag{163}$$

So

$$V_{BI} = -b|m_0|\frac{e^{3\phi}}{\mathrm{Vol}^{3/2}(\mathcal{M})} \tag{164}$$

where

$$b = 2\sqrt{2}\left|\frac{2}{3}(p_1 Z_1 - p_0 Z_0) + 6\sum_j Y_j P_j\right| \tag{165}$$

for both lattices.

The various contributions to V are homogeneous in $\mathrm{Vol}(T_k^2)$. Hence at the stationary point

$$0 = \sum_k \mathrm{Vol}(T_k^2)\frac{\partial V}{\partial \mathrm{Vol}(T_k^2)} = 6V_H + 7V_{F1} + 5V_{F2} + 3V_{m_0} + \frac{9}{2}V_{BI} \tag{166}$$

$$0 = \frac{\partial V}{\partial\phi} = 2V_H + 4V_{F1} + 4V_{F2} + 4V_{m_0} + 3V_{BI} \tag{167}$$

Eliminating V_{F1} gives

$$10V_H = 8V_{F2} + 16V_{m_0} + 3V_{BI} \tag{168}$$

Also, we require that $\partial V/\partial \mathrm{Vol}(T_k^2) = 0$, which gives

$$\begin{aligned} &|e_1 \mathrm{Vol}(T_1^2)|^2 = |e_3 \mathrm{Vol}(T_3^2)|^2 \equiv y^2 = \\ &= |e_2 \mathrm{Vol}(T_2^2)|^2 - 6\mathrm{Vol}(T_1^2)\mathrm{Vol}(T_3^2)\sum_j (E_j^2 + 3\hat{E}_j^2) + \frac{9\sqrt{3}|m_0|\mathrm{Vol}(\mathcal{M})^2}{4\mathrm{Vol}(T_1^2)\mathrm{Vol}(T_3^2)}\sum_j (|G_j| + 3|\hat{G}_j|) \end{aligned} \tag{169}$$

(with $y > 0$). It follows that

$$|e_2 \mathrm{Vol}(T_2^2)|^2 = y^2(1+\epsilon) - \frac{\eta}{y^2}\mathrm{Vol}(\mathcal{M})^2 \tag{170}$$

where

$$\epsilon \equiv \frac{6}{|e_1 e_3|}\sum_j (E_j^2 + 3\hat{E}_j^2) \tag{171}$$

$$\eta \equiv \frac{9\sqrt{3}|m_0 e_1 e_3|}{4}\sum_j (|G_j| + 3|\hat{G}_j|) \tag{172}$$

The requirement (153) that justifies the local treatment gives

$$|G_j, \hat{G}_j| \ll \frac{\sqrt{3}|m_0|y^2}{|e_1 e_3|} \tag{173}$$

so that

$$\eta \ll 27(m_0 y)^2 \tag{174}$$

We may also write $\mathrm{Vol}(\mathcal{M})$ in terms of y:

$$\mathrm{Vol}(\mathcal{M}) = \frac{1}{6}\prod_k \mathrm{Vol}(T_k^2) = \frac{y^3(1+\epsilon)^{1/2}}{(36|e_1 e_2 e_3|^2 + \eta y^2)^{1/2}} \tag{175}$$

so that

$$|e_2 \mathrm{Vol}(T_2^2)|^2 = \frac{y^2(1+\epsilon)}{1 + \frac{\eta y^2}{36|e_1 e_2 e_3|^2}} \tag{176}$$

Defining

$$x \equiv e^{\phi}\sqrt{\mathrm{Vol}(\mathcal{M})} \tag{177}$$

it follows from (168) that

$$10h = \left(16\mu m_0^2 + \frac{32\eta}{9y^2}\right)x^2 - 3b|m_0|x \tag{178}$$

which fixes x as a function of y. Hence

$$|m_0|x = \frac{3b}{32(\mu + \frac{2\eta}{9m_0^2 y^2})}\left(1 + \sqrt{1 + \frac{640(\mu + \frac{2\eta}{9m_0^2 y^2})h}{9b^2}}\right) \tag{179}$$

and at the stationary point, we may eliminate the dilaton and express the potential in terms of y alone:

$$V = \frac{A}{\mathrm{Vol}(\mathcal{M})^3} + \frac{B}{\mathrm{Vol}(\mathcal{M})^5} \tag{180}$$

where $\mathrm{Vol}(\mathcal{M})$ is given by (175) and

$$\begin{aligned} A &\equiv hx^2 + \mu m_0^2 x^4 - b|m_0|x^3 & (181) \\ B &\equiv \frac{2}{3}x^4 y^2(1+\epsilon)\left(1+\frac{\eta y^2}{3(36|e_1e_2e_3|^2+\eta y^2)}\right) & (182) \end{aligned}$$

with x given by (178). Since $B > 0$, it is easy to see that the potential $V \to +\infty$ as $y \to 0+$. Similarly, $V \to 0$ as $y \to \infty$. The limit is approached from above or below depending upon the sign of A in this region. If $A < 0$, then there is certainly an anti-de-Sitter minimum at a finite value of y; otherwise, no conclusion can be reached without a more detailed consideration of the parameters. It follows from (181) and (178) that

$$A \sim \frac{13}{3}x^2(h - \mu m_0^2 x^2) \quad \text{as} \quad y \to \infty \tag{183}$$

In the same limit, (179) gives

$$|m_0|x \simeq \frac{3b}{32\mu}\left(1+\sqrt{1+\frac{640\mu}{9b^2}}\right) \tag{184}$$

Then $A < 0$ if and only if

$$b^2 > 4\mu h \tag{185}$$

To proceed further, we need to know the dependence of the moduli $g_{0,1}, f_j$ that appear in (161) on $Z_{0,1}, Y_j$. For simplicity, we consider only the bulk contributions $g_{0,1}$ and assume that these derive from a homegeneous quadratic prepotential $\mathcal{G}$, defined in (162), of the form

$$\mathcal{G}(Z_0, Z_1) = \alpha Z_0^2 + 2\beta Z_0 Z_1 + \gamma Z_1^2 \tag{186}$$

with α, β and γ (real) constants (not functions of Z_0/Z_1). Then the moduli $g_{0,1}$ are given by

$$\begin{aligned} g_0 &= -\frac{\partial \mathcal{G}}{\partial Z_0} = -2(\alpha Z_0 + \beta Z_1) & (187) \\ g_1 &= \frac{\partial \mathcal{G}}{\partial Z_1} = 2(\beta Z_0 + \gamma Z_1) & (188) \end{aligned}$$

The question we address is whether $\mathcal{G}$ may be chosen so that (185) is always satisfied. Keeping only the bulk contributions, the minimum value of

$$b^2 = \frac{32}{9}(Z_0 p_0 - Z_1 p_1)^2 \tag{189}$$

subject to the constraint (161) that $\mathcal{G}(Z_0, Z_1) = 3/32$ is

$$b^2 = 3\frac{\gamma p_0^2 + 2\beta p_0 p_1 + \alpha p_1^2}{\alpha\gamma - \beta^2} \tag{190}$$

Evidently, we may ensure that (185) is satisfied by choosing α, β, γ sufficiently small. On the **AAA** lattice,

$$\frac{p_0}{p_1} = 3 + 2t_1^c = 5, 1 \tag{191}$$

so that the minimum value of b^2 and $2h$ are

$$\begin{aligned} b^2 &= \frac{3}{p_1^2(\alpha\gamma - \beta^2)}[15\gamma + 6\beta + \alpha + 4t_1^c(3\gamma + \beta)] & (192) \\ &= \frac{3}{p_1^2(\alpha\gamma - \beta^2)}(25\gamma + 10\beta + \alpha, \gamma + 2\beta + \alpha) & (193) \\ 2h &= \frac{8}{3p_1^2}(7 + 6t_1^c) & (194) \\ &= \frac{8}{3p_1^2}(13, 1) & (195) \end{aligned}$$

for $t_1^c = +1, -1$ respectively. On the **BAA** lattice, since $p_0 = p_1$ in this case, b^2 and $2h$ have the same values as in the $t_1^c = -1$ case for the **AAA** lattice.

The untwisted part of the 4-form flux F_4^{bg} given in equation (94). It is specified by the quantities $e_k \mathrm{Vol}_k / \mathrm{Vol}_6$ $(k = 1, 2, 3)$. Using (169), the ratios of the metric moduli $\gamma_i/\gamma_j = e_j \mathrm{Vol}_j / e_i \mathrm{Vol}_i$ are specified for a given value of F_4^{bg}. The minimisation of V fixes $y^2/|e_1 e_2 e_3|^2$, and F_4^{bg} also specifies the combination $|e_1 e_2 e_3|/\mathrm{Vol}_6^2$. Thus, the stabilisation fixes the overall scale of the metric moduli $(\gamma_1\gamma_2\gamma_3)^2 = (y^2/|e_1 e_2 e_3|)^3 (|e_1 e_2 e_3|/\mathrm{Vol}_6^2)$ in terms of the specified background fluxes. Similarly, the (untwisted) background flux H_3^{bg}, defined in equation (87), is specified by $p_{0,1}/\sqrt{\mathrm{Vol}_6}$. Thus equation (179) fixes $x/\sqrt{\mathrm{Vol}_6}$ in terms of the background fluxes m_0 and H_3^{bg}. With x defined in (177), it follows that $x/\sqrt{\mathrm{Vol}_6} \simeq e^{\phi}\sqrt{\gamma_1\gamma_2\gamma_3}$, and since the moduli $\gamma_{1,2,3}$ have already been fixed, this result stabilises the dilaton ϕ in terms of the background fluxes. The argument may be extended to include the twisted moduli.

4 Stability

Since we have taken $F_2^{bg} = 0$, the $|F_2|^2$ and $|F_4|^2$ terms in the the action S_{IIA}, given in (9), are at least quadratic in the fields B_2, there being no $\mathbb{Z}_6'$-invariant 1-form fields C_1. The Chern-Simons terms have already been accounted for in the minimisation of X. Thus the whole action S_{IIA} is at least quadratic in the moduli fields $b_k,\ B_j,\ \hat{B}_j$ defined in (44) ... (46) and (51) ... (53), and we may consistently set all of their expectation values to be zero, as in (113) in the supersymmetric case. However, there are fluctuations $b_k(x), B_j(x), \hat{B}_j(x)$ around this solution, and the $B_2 \wedge B_2 \wedge^* F_4^{bg}$ contribution to $|F_4|^2$ can make the solution unstable if the mass matrix for the fluctuations has a negative eigenvalue.

After eliminating the Lagrange multiplier $\mathcal{F}_0 \equiv dC_3$, the effective action deriving from this field is given in (131) with X in (132). With the B_2-moduli set to zero, the stabilised linear combination of the axions given in (102) reduces to

$$\frac{8}{3}(x_0 p_0 - x_1 p_1) - 24 \sum_j X_j P_j = e_0 \tag{196}$$

The $B_2 \wedge F_4^{bg} + C_3 \wedge H_3^{bg}$ piece in X is linear in the fluctuation fields and the above stabilised combination of axion fields. Hence the action (131) mixes them and we need to consider the quadratic terms, including kinetic terms, for both sets of fields simultaneously. The unstabilised (orthogonal) axion fields are, of course, massless.

The kinetic terms for the B_2 field fluctuations derive from the contribution

$$-\frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\frac{1}{2}\,e^{-2\phi}|H_3|^2 \quad \supset \quad -\frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}e^{-2\phi}\frac{1}{2}dB_2\wedge^* dB_2 \tag{197}$$

$$= \quad -\frac{1}{2\kappa_{10}^2}\int d^4x\,\sqrt{-g_E}\mathcal{L}_K(B) \tag{198}$$

where the kinetic Lagrangian density is

$$\mathcal{L}_K(B) = \frac{1}{2}\sum_{k=1,2,3}(\partial_\mu \tilde{b}_k)(\partial^\mu \tilde{b}_k) + \frac{1}{2}\sum_{j=1,4,5,6}[(\partial_\mu \tilde{B}_j)(\partial^\mu \tilde{B}_j) + (\partial_\mu \tilde{\hat{B}}_j)(\partial^\mu \tilde{\hat{B}}_j)] \tag{199}$$

with $\partial^\mu \tilde{b}_k = g_E^{\mu\nu}\partial_\nu \tilde{b}_k$ *etc.*, and the fields $\tilde{b}_k, \tilde{B}_j, \tilde{\hat{B}}_j$ defined so that they are canonically normalised:

$$\tilde{b}_k \equiv \frac{2b_k}{\mathrm{Vol}(T_k^2)} \tag{200}$$

$$\tilde{B}_j \equiv \sqrt{\frac{2\mathrm{Vol}(T_2^2)}{\mathrm{Vol}(\mathcal{M})}}B_j \tag{201}$$

$$\tilde{\hat{B}}_j \equiv \sqrt{\frac{6\mathrm{Vol}(T_2^2)}{\mathrm{Vol}(\mathcal{M})}}\hat{B}_j \tag{202}$$

Quadratic terms in these fields arise from

$$-\frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}[|F_2|^2+|F_4|^2] \quad \supset -\frac{1}{2\kappa_{10}^2}\int [m_0^2 B_2\wedge^* B_2 - m_0 B_2\wedge B_2\wedge^* F_4^{bg}]$$

$$= -\frac{1}{2\kappa_{10}^2}\int d^4x\sqrt{-g_E}\,\frac{e^{4\phi}}{\mathrm{Vol}^2(\mathcal{M})}\quad \left(\sum_k \left[m_0^2\mathrm{Vol}(\mathcal{M})\tilde{b}_k\tilde{b}_k + \frac{4}{3}m_0\tilde{b}_1\tilde{b}_2\tilde{b}_3\frac{e_k\mathrm{Vol}(T_k^2)}{\tilde{b}_k}\right]\right.$$

$$+\sum_j\left[m_0^2\mathrm{Vol}(\mathcal{M})-\frac{2}{3}m_0e_2\mathrm{Vol}(T_2^2)\right](\tilde{B}_j^2+\tilde{\hat{B}}_j^2)\quad +16m_0\sqrt{\frac{2\mathrm{Vol}(\mathcal{M})}{\mathrm{Vol}(T_2^2)}}\tilde{b}_2\sum_j(\tilde{B}_jE_j+\sqrt{3}\tilde{\hat{B}}_j\hat{E}_j)$$

$$\left.-4m_0\mathrm{Vol}(T_2^2)\tilde{b}_1\tilde{b}_3\sum_j(G_j+3\hat{G}_j)\quad -\frac{2m_0^2\mathrm{Vol}(\mathcal{M})}{\sqrt{3}}\sum_j\left[\tilde{B}_j^2s_j+\tilde{\hat{B}}_j^2\hat{s}_j\right]\right) \tag{203}$$

where $s_j,\hat{s}_j$ are respectively the signs of $G_j/m_0,\hat{G}_j/m_0$, and the last term follows when we substitute the stabilised values (151) and (152) of the blow-up volumes.

The kinetic terms for the C_3 fluctuations arise from

$$-\frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}|F_4|^2 \quad\supset\quad -\frac{1}{2\kappa_{10}^2}\int dC_3\wedge^* dC_3 \tag{204}$$

$$= -\frac{1}{2\kappa_{10}^2}\int d^4x\,\sqrt{-g_E}\,\frac{e^{2\phi}}{\mathrm{Vol}(\mathcal{M})}\quad\left[\frac{4}{3}(\partial(x_0-x_1))^2+\frac{4}{3}(\partial(x_0+x_1))^2+6\sum_j(\partial X_j)^2\right] \tag{205}$$

$$= -\frac{1}{2\kappa_{10}^2}\int d^4x\,\sqrt{-g_E}\quad \frac{1}{2}\left[(\partial\tilde{x}_1)^2+(\partial\tilde{x}_2)^2+\sum_j(\partial\tilde{X}_j)^2\right] \tag{206}$$

and the canonically normalised fields $\tilde{x}_{1,2}$ and $\tilde{X}_j$ are given by

$$\tilde{x}_1 \;\equiv\; \sqrt{\frac{2}{3\mathrm{Vol}(\mathcal{M})}}\,2e^{\phi}(x_0-x_1) \tag{207}$$

$$\tilde{x}_2 \;\equiv\; \sqrt{\frac{2}{3\mathrm{Vol}(\mathcal{M})}}\,2e^{\phi}(x_0+x_1) \tag{208}$$

$$\tilde{X}_j \;\equiv\; \sqrt{\frac{3}{\mathrm{Vol}(\mathcal{M})}}\,2e^{\phi}X_j \tag{209}$$

Quadratic terms in b_k and $x_{0,1}$ arise from (131)

$$S=-\frac{1}{2\kappa_{10}^2}\int X\wedge^* X\supset -\frac{1}{2\kappa_{10}^2}\int(B_2\wedge F_4^{bg}+C_3\wedge H_3^{bg})\wedge^*(B_2\wedge F_4^{bg}+C_3\wedge H_3^{bg}) \tag{210}$$

As noted previously, the only coupled combination of axion fields corresponds to the stabilised axion, whose normalised field $\tilde{a}$ is given in terms of the rescaled fields $\tilde{x}_{1,2},\tilde{X}_j$ by

$$p_0x_0-p_1x_1-9\sum_j P_jX_j\propto(p_0+p_1)\tilde{x}_1+(p_0-p_1)\tilde{x}_2-6\sqrt{2}\sum_j P_j\tilde{X}_j\equiv N\tilde{a} \tag{211}$$

where

$$N=\sqrt{2}\left[p_0^2+p_1^2+36\sum_j P_j^2\right]^{1/2} \tag{212}$$

We shall consider only the *untwisted* contibutions. Then the quadratic terms deriving from (210) are

$$S=-\frac{1}{2\kappa_{10}^2}\int d^4x\sqrt{-g_E}\,\frac{4e^{4\phi}}{9\mathrm{Vol}^3(\mathcal{M})}\left(\sum_k\mathrm{Vol}(T_k^2)\tilde{b}_ke_k+e^{-\phi}\sqrt{3\mathrm{Vol}(\mathcal{M})(p_0^2+p_1^2)}\tilde{a}\right)^2 \tag{213}$$

Stability requires that the eigenvalues of the mass matrix are all positive. However the uncoupled axion is massless, so the best we can hope for is that the remaining four mass eigenstates are non-tachyonic. The mass matrix deriving from (213) may be written in the form

$$\mathbf{m}^2 = \frac{2e^{4\phi}m_0^2}{\mathrm{Vol}(\mathcal{M})}\begin{pmatrix} 1+\alpha^2\gamma^2 & \alpha\gamma(s_3+s_1s_2\alpha) & \alpha(s_2+s_3s_1\alpha\gamma^2) & s_1\alpha\beta\gamma \\ \alpha\gamma(s_3+s_1s_2\alpha) & 1+\alpha^2 & \alpha\gamma(s_1+s_2s_3\alpha) & s_2\alpha\beta \\ \alpha(s_2+s_3s_1\alpha\gamma^2) & \alpha\gamma(s_1+s_2s_3\alpha) & 1+\alpha^2\gamma^2 & s_3\alpha\beta\gamma \\ s_1\alpha\beta\gamma & s_2\alpha\beta & s_3\alpha\beta\gamma & \beta^2 \end{pmatrix} \tag{214}$$

where

$$\alpha \equiv \frac{4|e_1e_2e_3|}{|m_0|y^2} \tag{215}$$

$$\beta \equiv \frac{2(p_0^2+p_1^2)^{1/2}}{\sqrt{3}m_0x} \tag{216}$$

$$\gamma \equiv \left(1+\frac{\eta y^2}{36|e_1e_2e_3|^2}\right)^{1/2}(1+\epsilon)^{-1/2} \tag{217}$$

and $s_{1,2,3} = \pm 1$ are the signs of $e_{1,2,3}$. The general expressions for the eigenvalues are too large to be tractable, but positive-definiteness is ensured provided that the following quantities are all positive:

$$\mathrm{tr}(\mathbf{m}^2) = \sum_i (m^2)_i = \beta^2 + 3 + a^2(1+2\gamma^2) \tag{218}$$

$$\det(\mathbf{m}^2) = \prod_i (m^2)_i = \beta^2(1-a^2+2a^3\gamma^2-2a^2\gamma^2) \equiv d(a) \tag{219}$$

$$\sum_{i,j}(m^2)_i(m^2)_j = 3\beta^2+6a^3\gamma^2+2a^2\gamma^2+3+a^2 \equiv d_4(a) \tag{220}$$

$$\sum_{i,j,k}(m^2)_i(m^2)_j(m^2)_k = 3\beta^2+1+4a^4\gamma^2+4a^3\gamma^2-a^4-2a^2\beta^2\gamma^2-a^2\beta^2 \equiv d_6(a) \tag{221}$$

where

$$a \equiv s_1s_2s_3\alpha \tag{222}$$

When $\gamma^2 > 1/4$ it is obvious that for large, positive values of $a \gg 1$ all of these *are* positive. The question is whether there are other values, in particular negative values, for which we also have positive-definiteness, and what can be said when $\gamma^2 \leq 1/4$. By inspection it is clear that the trace is automatically positive. For general (non-zero) values of β and γ, $\det(\mathbf{m}^2) = d(a) > 0$ provided that

$$a_1 \equiv \frac{1}{4\gamma^2}\left(1-\sqrt{1+8\gamma^2}\right) < a < a_2 \tag{223}$$

$$\text{or} \qquad a_3 < a \tag{224}$$

where

$$a_2 = \frac{1}{4\gamma^2}\left(1+\sqrt{1+8\gamma^2}\right), \quad a_3 = 1 \quad \text{for} \quad \gamma^2 > 1 \tag{225}$$

$$a_2 = 1, \quad a_3 = \frac{1}{4\gamma^2}\left(1+\sqrt{1+8\gamma^2}\right) \quad \text{for} \quad \gamma^2 < 1 \tag{226}$$

Note that a_1 is always negative, and $a_{2,3}$ positive. In the special case that $\gamma = 0$, the function $d(a) = \beta^2(1-a^2)$ is positive only in the range $-1 < a < 1$.

We also require that $d_4(a)$ is positive. Evidently this is always the case for $a > 0$, so we need only consider whether negative values of a lead to stronger constraints than those already derived. According to (223), the most negative value that we need to consider is $a = a_1$, which satisfies $d(a_1) = 0$. For this value of a it follows that

$$d_4(a_1) = 3\beta^2 + 4a_1^2(1+2\gamma^2) > 0 \tag{227}$$

Further, $d_4(a)$ has only a single real (negative) root, so the positivity of $\sum_{i,j}(m^2)_i(m^2)_j$ gives no further constraints.

Finally, we require also that $d_6(a)$ is positive. It is convenient to write

$$d_6(a) = N(a) - \beta^2 D(a) \tag{228}$$

where

$$\begin{aligned} N(a) &\equiv (4\gamma^2-1)a^4 + 4\gamma^2 a^3 + 1 &(229)\\ &= (a+1)[(4\gamma^2-1)a^3 + a^2 - a + 1] &(230)\\ D(a) &\equiv (2\gamma^2+1)a^2 - 3 &(231) \end{aligned}$$

The special case in which $\gamma = 0$ is easy to analyse. In this case $N(a) = 1 - a^4$ is positive only in the range $-1 < a < 1$ in which $d(a)$ is also positive. Since $D(a) = a^2 - 3$ is negative throughout this range, it is only for values of a in this range that we have positive definiteness. The case in which $\gamma^2 = 1/4$ is also easy to analyse. Positivity of $d(a)$ requires that either $1 - \sqrt{3} = a_1 < a < a_2 = 1$ or $a > 1 + \sqrt{3}$. The function $N(a) = 1 + a^3$ is positive only when $a > -1$. Thus $N(a)$ is positive in both of these ranges, while $D(a) = 3(a^2-2)/2$ is negative in the region $a_1 < a < a_2$, but positive in $a > a_3$. It follows that $\mathbf{m^2}$ is positive definite for any value of β^2 when a is in the range $a_1 < a < a_2$, but only for values of $\beta^2 < N(a)/D(a)$ in the range $a > a_3$.

The full analysis of the conditions in which $d_6(a)$ and $d(a)$ are both positive for general values of γ^2 is given in Appendix B. The conclusions are as follows: For values of a in the range $a_1 < a < a_2$, the mass matrix $\mathbf{m}^2$ is positive definite for all values of γ^2 and all values of β^2. If $0.1955 \lesssim \gamma^2 < 1/4$, there is in addition a finite region $a_3 < a < a_5$ in which $\mathbf{m}^2$ is positive definite but only for values of β^2 that are bounded above by $N(a)/D(a)$. Finally, if $\gamma^2 > 1/4$, there is an infinite region $a > a_3$ in which $\mathbf{m}^2$ is positive definite, again for values of β^2 that are bounded above by $N(a)/D(a)$. Here $a_{1,2,3}$, defined in equations (223), (225) and (226), specify the regions in which $d(a) > 0$, and a_5 is the root of the cubic factor in equation (230).

Although we have been discussing the conditions under which the (untwisted) mass eigenstates are non-tachyonic, in principle this is too strong a requirement in the anti-de Sitter space of our vacuum solutions. Tachyonic mass eigenstates are stable provided that they satisfy the Breitenlohner-Freedman bound [32, 33]

$$m_i^2 \geq m_{BF}^2 \equiv -\frac{3}{4}|V_{\min}| \tag{232}$$

where $-|V_{\min}|$ is the value of the potential at the anti-de Sitter minimum. The massless uncoupled axion obviously satisfies the bound, so it will not generate instability. However, determining which values of a lead to other mass eigenstates that satisfy this weaker constraint is something that can only be done when $V_{\min}$ has actually been calculated, and this in turn requires a detailed consideration of the parameters, as already noted. The expectation or, more accurately, the hope is that when the anti-de Sitter minimum is lifted to Minkowski, in the manner of KKLT [34], then the tachyonic states will be lifted too. However, as Conlon has noted [35], it is not clear that *all* tachyons will be lifted by this mechanism. The uplifting is generally rather poorly controlled, and it is at least plausible that there may remain tachyons in the Minkowski space.

5 E2-instantons and Yukawa couplings

We have so far fixed only *one* linear combination of the axion fields. As noted previously, we may use non-perturbative effects to stabilise the remaining axions. The non-perturbative effects under discussion are Dp-branes that wrap non-trivial cycles in the compactification space $\mathcal{M}_6$, and that are pointlike in $\mathcal{M}_4$. Their world-volume is $(p+1)$-dimensional and spacelike, so they are Euclidean Dp-branes, called Ep-branes or Ep-instantons for short. In type IIA string theory, p is even and $p+1 \leq 6$. Hence $p = 0, 2, 4$. Since there are no non-trivial 1- and 5-cycles on the orientifold $T^6/\mathbb{Z}_6'$ with which we are concerned, only E2-branes are relevant. The instanton action S_{inst} is given by [36]

$$S_{\text{inst}} = 2\pi \left(\frac{1}{g_s} \int_{\Xi} \mathrm{Re}\, \Omega_3 - i \int_{\Xi} C_3 \right) \tag{233}$$

where Ξ is the 3-cycle wrapped by the E2-brane, and Ω_3 is the holomorphic 3-form. Evidently an E2-instanton is coupled to the axion fields in C_3 and can lift (some of) the degeneracy of the axions that are not stabilised by the background flux $m_0 H_3^{bg}$. Quite generally, we may expand Ξ in terms of the untwisted 3-cycles ρ_p $(p = 1, 3, 4, 6)$ and the exceptional 3-cycles $\epsilon_j, \tilde{\epsilon}_j$ $(j = 1, 4, 5, 6)$, so that

$$\Xi = \frac{1}{2}\sum_p Z_p \rho_p + \frac{1}{2}\sum_j (z_j \epsilon_j + \tilde{z}_j \tilde{\epsilon}_j) \tag{234}$$

where $Z_p, z_j, \tilde{z}_j$ are integers. Supersymmetry constrains these coefficients. On the **AAA** lattice, it requires that

$$3Z_3 - 2Z_4 + Z_6 = 0 \tag{235}$$

$$2Z_1 - Z_3 + Z_6 > 0 \tag{236}$$

As displayed in equations (189) ... (194) of reference [12], there are three types of supersymmetric 3-cycle:

$$\begin{aligned} Z_p &= (1,0,0,0) \bmod 2 \quad (n_k, m_k) = (1,0;1,0;1,0) \bmod 2 && (237)\\ &= (1,\theta,1,\theta) \bmod 2 \quad (n_k, m_k) = (1,1;\theta,1;1,1) \bmod 2 && (238)\\ &= (0,0,1,0) \bmod 2 \quad (n_k, m_k) = (0,1;1,1;0,1) \bmod 2 && (239) \end{aligned}$$

(with $\theta = 0, 1$) called respectively c-, d_θ- and e-type. They are associated with exceptional parts having

$$\begin{aligned} c: &\quad (z_{1,4}; \tilde{z}_{1,4}) \text{ or } (z_{5,6}; \tilde{z}_{5,6}) = (0,0;1,1) \bmod 2 && (240)\\ d_\theta: &\quad (z_{1,6}; \tilde{z}_{1,6}) \text{ or } (z_{4,5}; \tilde{z}_{4,5}) = (\theta,\theta;1,1) \bmod 2 && (241)\\ e: &\quad (z_{1,5}; \tilde{z}_{1,5}) \text{ or } (z_{4,6}; \tilde{z}_{4,6}) = (0,0;1,1) \bmod 2 && (242) \end{aligned}$$

With the general form of the instanton's 3-cycle Ξ given in equation (234), and with C_3 on the **AAA** lattice given in equation (75), we find

$$\begin{aligned} \text{Im } S_{\text{inst}} &= -6\pi \left(\frac{2}{3^{1/4}}[(2Z_1 - Z_3)(x_0 + x_1) + Z_6(x_0 - x_1)] + \frac{3^{1/4}}{\sqrt{2}} \sum_j X_j z_j \right) && (243)\\ &\propto 2[(2Z_1 - Z_3) r^{1/4} \tilde{x}_2 + Z_6 r^{-1/4} \tilde{x}_1] + \frac{1}{\sqrt{3}} \sum_j \tilde{X}_j z_j && (244) \end{aligned}$$

using the canonically normalised fields defined in equations (207), (208) and (209). In general, this is quite different from the combination a given in equation (211) that is stabilised by the background flux. Evidently a separate instanton is required for each of the unstabilised axions.

Similarly, on the **BAA** lattice, supersymmetry requires that

$$2Z_3 - Z_1 - 3Z_4 = 0 \tag{245}$$

$$Z_1 - Z_4 + 2Z_6 > 0 \tag{246}$$

and, as displayed in equations (275) ... (280) of reference [12], again there are three types of supersymmetric 3-cycle:

$$\begin{aligned} Z_p &= (0,1,0,0) \bmod 2 \quad (n_k, m_k) = (1,0;1,1;1,0) \bmod 2 && (247)\\ &= (\theta,1,\theta,1) \bmod 2 \quad (n_k, m_k) = (1,1;1,\theta;1,1) \bmod 2 && (248)\\ &= (0,0,0,1) \bmod 2 \quad (n_k, m_k) = (0,1;0,1;0,1) \bmod 2 && (249) \end{aligned}$$

(with $\theta = 0, 1$) called respectively c-, d_θ- and e-type. They are associated with exceptional parts having

$$\begin{aligned} c: &\quad (z_{1,4}; \tilde{z}_{1,4}) \text{ or } (z_{5,6}; \tilde{z}_{5,6}) = (1,1;0,0) \bmod 2 && (250)\\ d_\theta: &\quad (z_{1,6}; \tilde{z}_{1,6}) \text{ or } (z_{4,5}; \tilde{z}_{4,5}) = (1,1:\theta,\theta) \bmod 2 && (251)\\ e: &\quad (z_{1,5}; \tilde{z}_{1,5}) \text{ or } (z_{4,6}; \tilde{z}_{4,6}) = (1,1;0,0) \bmod 2 && (252) \end{aligned}$$

In this case, we find that

$$\mathrm{Im}\, S_{\mathrm{inst}} = -6\pi R_3 \left(\frac{2}{3^{1/4}}[Z_1(x_0+x_1) - (Z_4-2Z_6)Z_6(x_0-x_1)] - \frac{3^{1/4}}{\sqrt{2}} \sum_j X_j \tilde{z}_j \right) \quad (253)$$

$$\propto 2[Z_1 r^{1/4}\tilde{x}_2 - (Z_4-2Z_6)r^{-1/4}\tilde{x}_1] - \frac{1}{\sqrt{3}} \sum_j \tilde{X}_j \tilde{z}_j \quad (254)$$

Again, this is generally quite different from the combination a given in equation (211).

As noted in the Introduction, the surviving global $U(1)$ symmetries in our models forbid some of the Yukawa couplings that are needed to give non-zero masses to the quarks and leptons via the Higgs mechanism. Consider, for example, the model described by the fourth solution in Table 1 of reference [12]. The weak hypercharge Y is a linear combination of the $U(1)$ charges $Q_{a,c,d}$ associated respectively with the $SU(3)_c$ stack a, and the $U(1)$ stacks c, d.

$$Y = \frac{1}{6}Q_a + y_c Q_c + \frac{1}{2}Q_d \quad (255)$$

where $y_c = \pm\frac{1}{2}$. Using equations (63) and (66) of that paper, the intersection numbers of a with the $SU(2)_L$ stack b and its orientifold image b' are given by

$$(a \cap b, a \cap b') = (1,2) \quad \text{if} \quad (-1)^{\tau_0^a+\tau_0^b} = 1 \quad (256)$$

$$= (2,1) \quad \text{if} \quad (-1)^{\tau_0^a+\tau_0^b} = -1 \quad (257)$$

thereby generating the required total of $3Q_L$ quark doublets (with $Y = \frac{1}{6}$). Similarly, using equation (247), the $U(1)$ stack d and its orientifold image d' have intersection numbers

$$(a \cap d, a \cap d') = (0,0) \quad (258)$$

so that there are no quark-singlet states q_L^c at these intersections. Choosing $y_c = -\frac{1}{2}$ in (255), the Higgs doublet H_u with $Y = \frac{1}{2}$ arises at the intersection of b with the $U(1)$ stack c, while H_d with $Y = -\frac{1}{2}$ arises at the intersection with its orientifold image c':

$$(b \cap c, b \cap c') = (1,1) \quad (259)$$

The quark singlets arise at the intersections of a with c and c'

$$(c \cap a, c' \cap a) = (3,3) \quad (260)$$

the former giving $3u_L^c$ and the latter $3d_L^c$. First, consider the case described by (257). u-quark mass terms arising from the two Q_L states at $a \cap b$ require the coupling of the states at $a \cap b$, $b \cap c$ and $c \cap a$, which is allowed by the conservation of Q_a, Q_b and Q_c. However, the u-quark mass term arising from the Q_L state at $a \cap b'$ requires the coupling of the states at $a \cap b'$, $b \cap c$ and $c \cap a$, which is allowed by the conservation of Q_a and Q_c, but *not* by Q_b, since the product has $\Delta Q_b = 2$. Similarly, only two d-quark mass terms are allowed by conservation of Q_b. The alternative choice described by (256) allows only one quark mass term for both u- and d-type quarks.

We also have

$$(d' \cap b, d' \cap b') = (1,2) \qquad \text{if} \quad (-1)^{\tau_0^b+\tau_0^d}\chi = -1 = (-1)^{\tau_0^a+\tau_0^b} \quad (261)$$

$$= (2,1) \qquad \text{if} \quad (-1)^{\tau_0^b+\tau_0^d}\chi = 1 = (-1)^{\tau_0^a+\tau_0^b} \quad (262)$$

which generate the required total of $3L$ lepton doublets (with $Y = -\frac{1}{2}$), while the lepton singlets arise from

$$(c' \cap d', c \cap d') = (3,3) \quad (263)$$

the former giving the $3\ell_L^c$ charged lepton singlets, and the latter the $3\nu_L^c$ the neutrino singlet states. For the case (257) under consideration, equation (261) gives the location of the lepton doublets. The charged

lepton mass term arising from the lepton doublet at $d' \cap b$ require the coupling of the states at $d' \cap b$, $b \cap c'$ and $c' \cap d'$, which is allowed by the conservation of Q_b, Q_c and Q_d. However, the charged lepton mass terms arising from the two lepton doublets at $d' \cap b'$ require couplings that again have $\Delta Q_b = 2$. Similarly, only one neutrino mass term is allowed by conservation of Q_b. The alternative choice described by (262) allows two lepton mass term for both charged leptons and neutrinos. Thus, at the perturbative level, after electroweak symmetry breaking, we either have two massive quark generations and one massive lepton generation, or *vice versa*. In the model discussed in reference [11], the same correlation is obtained.

In both cases the missing couplings can only be provided by non-perturbative instanton effects. These generate terms in the superpotential W of the form

$$W \simeq \prod_i \Phi_i e^{-S_{\text{inst}}} \tag{264}$$

that violate the global $U(1)$ symmetries that survive after the Green-Schwarz mechanism breaks any anomalous $U(1)$ gauge symmetry [36]; here Φ_i are the (generally charged) matter superfields and S_{inst} is the action of the non-perturbative instanton. Such a term *is* allowed if and only if the gauge transformation of the matter field product $\prod_i \Phi_i$ under an anomalous $U(1)$ gauge transformation is cancelled by the transformation of the exponential factor induced by the *shift* of $\text{Im } S_{\text{inst}}$ under the $U(1)$ transformation [37]. Under a $U(1)_\kappa$ gauge transformation, associated with the stack κ, parametrised by Λ_κ, in which the 1-form vector potential A_1^κ is shifted by

$$\delta A_1^\kappa = d\Lambda_\kappa \tag{265}$$

the imaginary part $\text{Im } S_{\text{inst}}$ of the instanton action (233) is shifted by

$$\delta \left(\text{Im } S_{\text{E2}}\right) = \Lambda_\kappa Q_\kappa(E_2) \tag{266}$$

where $Q_\kappa(E_2)$ is the $U(1)_\kappa$ charge of the instanton, given by

$$Q_\kappa(E_2) = -\Xi \cap N_\kappa(\kappa - \kappa') \tag{267}$$

To repair the missing Yukawa couplings we require that

$$\begin{aligned} Q_b(E_2) &= -2 && (268) \\ Q_a(E_2) = 0 &= Q_c(E_2) = Q_d(E_2) && (269) \end{aligned}$$

The general form of Ξ is given in equation (234) Then, using our solution for the $SU(2)_L$ stack on the **AAA** lattice given in Table 1 and equation (66) of reference [12], it follows from (268) above that

$$(-1)^{\tau_0^b+1}[z_1 + (-1)^{\tau_2^b} z_5] = 1 \tag{270}$$

so that z_1 or z_5, but not both, are odd. We also require, as in (269), that the instanton has zero charge with respect to the other $U(1)$ charges. For Q_c this is guaranteed, since $c = c'$. Further, since $a - a' = d' - d$ in our solution, $Q_a(E2) = 0$ ensures that $Q_d(E2) = 0$. Thus, there is just one further constraint, which yields

$$2Z_1 - Z_3 - Z_6 + (-1)^{\tau_0^a}[z_1 + (-1)^{\tau_2^a} z_6] = 0 \tag{271}$$

It follows from (270) that Ξ is of d_1-type, as defined in equation (241), and it is easy to find solutions with all of the desired properties. For example

$$\Xi = \frac{1}{2}(\rho_1 - \rho_3 - \rho_4 + \rho_6) + \frac{1}{2}(-1)^{\tau_0^a+1}\left[\epsilon_1 + (-1)^{\tau_2^a}\epsilon_6 - \tilde{\epsilon}_1 - (-1)^{\tau_2^a}\tilde{\epsilon}_6\right] \tag{272}$$

with

$$\tau_0^a = \tau_0^b \bmod 2 \tag{273}$$

The above solution gives

$$(\Xi \cap b, \Xi \cap b') = (-1, -2) \tag{274}$$

Thus the required total instanton charge (268) derives from one (massless) particle at the intersection of Ξ with b, and two at the intersections of Ξ with b'. To repair the missing u-quark Yukawa, for example, we need a 5-point coupling in which both b and b' intersect the fractional 3-cycle Ξ of the instanton:

$$(a \cap b')(b' \cap \Xi)(\Xi \cap b)(b \cap c)(c \cap a) \tag{275}$$

Since $\Xi \cap b = -1$, we should interpret it as one intersection with $Q_b = +1$, rather than -1 intersections with $Q_b = -1$. However, since $b' \cap \Xi = 2$ is positive, the coupling (275) does *not* then conserve Q_b, and we conclude that we can*not* repair the Yukawa with this E2-instanton. Further, equation (268) requires that $\Xi \cap b - \Xi \cap b' = 1$ which can only be satisfied with non-zero $\Xi \cap b$ and $\Xi \cap b'$ when they have the *same* sign, as in the above solution. Consequently $\Xi \cap b$ and $b' \cap \Xi$ cannot have the same sign in any of the solutions, and they therefore contribute zero to the total Q_b charge in (275). Hence we cannot repair the Yukawa with any of the single E2-instanton solutions of the constraints. The same conclusion follows for the model discussed in reference [11], as well as for the models on the **BAA** lattice given in Table 6 of reference [12].

6 Conclusions

All of the models that we have considered have the attractive feature that they have the spectrum of the supersymmetric Standard Model, including a single pair of Higgs doublets, plus three right-chiral neutrino singlets. In the presence of the previously derived non-zero background field strength $m_0 H_3^{bg}$ they are also free of RR tadpoles, and therefore constitute consistent string-theory models. We showed in §2 that this background field also stabilises *one* of the axion moduli. Further, we found that it is easy to choose a non-zero background field strength F_4^{bg} that stabilises the Kähler and complex-structure moduli associated with the supersymmetric mininima at values within the Kähler cone in which the supergravity approximation is valid. In §3 we showed that there are also *non*-supersymmetric stationary points of the effective potential, and in §4 we determined the parameter ranges in which these are stable minima. The stabilisation of *all* of the axion moduli can only be achieved by the use of non-perturbative instanton effects, and these were discussed in §5. In principle, such effects might also restore the missing quark and lepton Yukawa couplings to the Higgs doublets that are needed to generate masses when the electroweak symmetry is spontaneously broken. However, we also showed that this does not happen for the particular models of interest here.

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Appendix A: The Freed-Witten anomaly

We need to assess whether any of our stacks $\kappa = a, b, c, d$ gives a non zero Freed-Witten anomaly

$$\Delta_\kappa \equiv H_3^{bg} \wedge [\kappa] \tag{276}$$

where the background flux H_3^{bg} in general has the form given in (87), and $[\kappa]$ is the 3-form that is the Poincaré dual of the fractional 3-cycle κ. The general form for the bulk part is

$$[\Pi_\kappa^{\mathrm{bulk}}] = \sum_p A_p^\kappa \eta_p \tag{277}$$

where η_p $(p = 1, 3, 4, 6)$ are the 3-forms that are the Poincaré duals of the bulk 3-cycles ρ_p, given in eqns (329) ... (332) of reference [12]. As noted previously, we need only consider the $\mathcal{R}$-even part of $[\kappa]$. On

the **AAA** lattice,

$$[\Pi_\kappa^{\rm bulk}]+[\Pi_\kappa^{\rm bulk}]' = A_3^\kappa(\eta_1+2\eta_3)+(2A_4^\kappa-A_6^\kappa)\eta_4 \quad (278)$$

$$= \frac{6}{R_1R_3R_5}[3A_3^\kappa(\sigma_0-\sigma_1-\sigma_2+\sigma_3)-(2A_4^\kappa-A_6^\kappa)(\sigma_0+\sigma_1+\sigma_2+\sigma_3)] \quad (279)$$

where the terms on the right-hand side are defined in eqns (55) ... (58). Then

$$H_3^{bg}\wedge[\Pi_\kappa^{\rm bulk}] = -\frac{6i}{R_1R_3R_5\sqrt{{\rm Vol}_6}}[3A_3^\kappa(p_0+p_1)+(2A_4^\kappa-A_6^\kappa)(p_1-p_0)]w_1\wedge w_2\wedge w_3 \quad (280)$$

where the (1,1)-forms w_k $(k=1,2,3)$ are defined in eqn (34). Similarly, the general form for the exceptional part is

$$[\Pi_\kappa^{\rm ex}] = \sum_j(\alpha_j^\kappa\chi_j+\tilde{\alpha}_j^\kappa\tilde{\chi}_j) \quad (281)$$

where $\chi_j,\tilde{\chi}_j$ $(j=1,4,5,6)$ are the Poincaré duals of the exceptional 3-cycles $\epsilon_j,\tilde{\epsilon}_j$; these are defined in eqns (369) and (370) of reference [12]. Then, on the **AAA** lattice,

$$[\Pi_\kappa^{\rm ex}]+[\Pi_\kappa^{\rm ex}]' = \sum_j(2\tilde{\alpha}_j^\kappa-\alpha_j^\kappa)\tilde{\chi}_j \quad (282)$$

$$= \sum_j\frac{R_3}{2\pi\hat{\lambda}_j^2\alpha{\rm Vol}_2}(2\tilde{\alpha}_j^\kappa-\alpha_j^\kappa)(\omega_j-\alpha\tilde{\omega}_j) \quad (283)$$

where ω_j and $\tilde{\omega}_j$ are defined in eqns (61) and (62). It follows that

$$H_3^{bg}\wedge[\Pi_\kappa^{\rm ex}] = \sum_{i,j}\frac{iR_3}{2\pi^2\hat{\lambda}_j^4{\rm Vol}_2^{3/2}}(2\tilde{\alpha}_j^\kappa-\alpha_j^\kappa)P_je_{(i,j)}\wedge e_{(i,j)}\wedge w_2 \quad (284)$$

Here $e_{(i,j)}$ are the localised (1,1)-forms defined in (35).

Consider first the solution on the **AAA** lattice given in §5.1 of reference [12], derived from the fourth solution in Table 1 of that paper. For the bulk parts of $\kappa=a,b,c,d$ respectively we have

$$3A_3^\kappa(p_0+p_1)+(2A_4^\kappa-A_6^\kappa)(p_1-p_0)=(6p_1,0,0,-6p_1) \quad (285)$$

so that cancellation of (the bulk part of) the Freed-Witten anomaly $\Delta_\kappa^{\rm bulk}$ for the $\kappa=a$ and d stacks requires that $p_1=0$. It follows from (88) that this in turn requires that $n_3=3n_6$ and it is evident from the discussion following eqn (88) that this is *not* satisfied by any of our solutions. Thus $\Delta_\kappa^{\rm bulk}\neq 0$ for the stacks $\kappa=a$ and d. Correspondingly, for the exceptional parts we find

$$2\tilde{\alpha}_j^\kappa-\alpha_j^\kappa = -3t_0^a(1,0,0,t_2^a) \quad (286)$$

$$= 2t_0^b(1,0,t_2^b,0) \quad (287)$$

$$= (0,0,0,0) \quad (288)$$

$$= 3t_0^a(1,0,0,t_2^a) \quad (289)$$

The localisation of the (1,1)-forms $e_{(i,j)}$ means that the cancellation of (the exceptional part of) the Freed-Witten anomaly $\Delta_\kappa^{\rm ex}$ for $\kappa=a,b$ and d requires that $P_1=0=P_5=P_6$, and hence that $\hat{n}_1=0=\hat{n}_5=\hat{n}_6$. We have already noted that tadpole cancellation requires that $|n_0\hat{n}_j|=12$ for $j=4,6$ so the last of these can*not* be satisfied; with the choice $t_1^c=1$, though, we can satisfy the first of these. Thus, $\Delta_\kappa^{\rm ex}\neq 0$ at least for the stacks $\kappa=a$ and d. A very similar analysis, with the same conclusions, applies to the other solutions derived from Table 1. The solutions on the **BAA** lattice are discussed in §6.2, and derive from Table 6 of reference [12]. We again find in all cases that $\Delta_\kappa^{\rm bulk}\neq 0$ for the stacks $\kappa=a$ and d.

Appendix B: Positive definiteness of the axionic fluctuations

It is clear from its definition that $D(a) \equiv (2\gamma^2+1)a^2 - 3$ is negative for $a_- < a < a_+$, where

$$a_\pm \equiv \pm\sqrt{\frac{3}{2\gamma^2+1}} \tag{290}$$

and positive elsewhere. For future information, it is easy to verify that

$$a_1 \equiv \frac{1}{4\gamma^2}\left(1-\sqrt{1+8\gamma^2}\right) \quad > \quad a_- \quad \forall\gamma \tag{291}$$

$$\frac{1}{4\gamma^2}\left(1+\sqrt{1+8\gamma^2}\right) \quad > \quad a_+ \quad \text{for} \quad \gamma^2<1 \tag{292}$$

$$\frac{1}{4\gamma^2}\left(1+\sqrt{1+8\gamma^2}\right) \quad < \quad a_+ \quad \text{for} \quad \gamma^2>1 \tag{293}$$

The analysis of $N(a) = (4\gamma^2-1)a^4 + 4\gamma^2 a^3 + 1$, defined in equation (229), is more complicated. For all values of γ^2 it has a root at $a = -1$, a saddle point at $a = 0$, and one other stationary point at

$$a = a_D \equiv \frac{3\gamma^2}{1-4\gamma^2} \tag{294}$$

When $0 < \gamma^2 < 1/4$ this stationary point is at a positive value of a and is a maximum. In this case, $N(a)$ is positive for $a_4 < a < a_5$, and negative elsewhere; here $-1 = a_4 < a_1 < 0$ and $a_5 > a_D > 0$ is the (positive) root $\alpha(\gamma^2)$ of the cubic factor in equation (230); in the special case $\gamma = 0$, for example, $a_5 = 1$. It is easy to verify that $N(a_-) \leq 0$ (actually for all values of γ^2, with equality only when $\gamma^2 = 1$); thus $a_- < a_4 < a_1$. Further, $N(a_+)$ is negative for $0 < \gamma^2 \lesssim 0.1255$, vanishes when $\gamma^2 \simeq 0.1255$, and is positive for all other values of γ^2; it follows that $a_5 < a_+$ for $0 < \gamma^2 \lesssim 0.1255$, but $a_+ < a_5$ for $0.1255 \lesssim \gamma^2 < 1/4$. Alternatively, when $\gamma^2 > 1/4$, a_D is negative and $N(a_D)$ is a minimum. In this case, $N(a)$ is negative for $a_4 < a < a_5$, and positive elsewhere, and now both a_4 and a_5 are negative roots of $N(a)$, with $a_4 < a_D < a_5$; for $\gamma^2 < 1$ the position of the stationary point satisfies $a_D < -1$, whereas for $\gamma^2 > 1$ we find $a_D > -1$. Thus when $\gamma^2 < 1$, $-1 = a_5 < a_1$ and a_4 is the root of the cubic in equation (230), whereas for $\gamma^2 > 1$, $-1 = a_4 < a_1$ and a_5 is the root of the cubic. Obviously, $a_+ > a_5$ for values of γ^2 in this range. These considerations lead us to consider three ranges of values for γ^2, with the signs of the functions given in the associated Tables.

- $0 < \gamma^2 \lesssim 0.1255$

Region	a	$N(a)$	$D(a)$	$d_6(a)$
I	$a < a_-$	-	+	-
II	$a_- < a < -1$	-	-	
III	$-1 < a < a_5$	+	-	+
IV	$a_5 < a < a_+$	-	-	
V	$a_+ < a$	-	+	-

Table 1: Signs of the functions $N(a), D(a), d_6(a)$ when $0 < \gamma^2 < 0.1255$

We need to identify the regions in which *both* $d(a)$ and $d_6(a)$ are positive. With a_1 defined in equation (223), we note that $D(a_1)$ is negative for any value of γ^2. Similarly, $N(a_1)$ is positive (actually for any value of γ^2). It follows that a_1 is in region III of Table 1; this is consistent with the observation above that $a_4 < a_1$ in this case. From equation (226) we see that $a_2 = 1$ for values of γ^2 in this range. Since $D(1) = 2(\gamma^2-1) < 0$, in this case, and $N(1) = 8\gamma^2 > 0$, it follows that

a_2 is also in region III of Table 1. Finally, using the value of a_3 given in (226), we find that $D(a_3)$ is positive, (actually for any $\gamma^2 < 1$; it vanishes when $\gamma^2 = 1$, and is negative for all other values.) For values of $\gamma^2 \lesssim 0.1915$, $N(a_3)$ is negative; (it vanishes when $\gamma^2 \simeq 0.1955$, and is positive for all other values.) It follows that a_3 is in region V of Table 1 in which $d_6(a)$ is negative. Thus the only region in which both $d(a)$ and $d_6(a)$ are positive is $a_1 < a < a_2 = 1$, and this is the case for all values of β^2; note that this range does not require the solution of the cubic.

- $0.1255 < \gamma^2 < \frac{1}{4}$

  The properties of the functions given above show that in this case a_1 is in region III of Table 2, as is a_2, and if $\gamma^2 \lesssim 0.1955$ then a_3 is in region V; otherwise it is region IV. Thus, if $\gamma^2 \lesssim 0.1955$, the only region in which both $d(a)$ and $d_6(a)$ are positive is again $a_1 < a < a_2 = 1$, and as before this is the case for all values of β^2. However, in the case $\gamma^2 \gtrsim 0.1955$, $N(a)$ and $D(a)$ are both positive in the region $a_3 < a < a_5$, so that both $d(a)$ and $d_6(a)$ are positive here too, provided that $\beta^2 < N(a)/D(a)$. The determination of a_5 requires the solution of the cubic, which we discuss below.

Region	a	$N(a)$	$D(a)$	$d_6(a)$
I	$a < a_-$	-	+	-
II	$a_- < a < -1$	-	-	
III	$-1 < a < a_+$	+	-	+
IV	$a_+ < a < a_5$	+	+	
V	$a_5 < a$	-	+	-

Table 2: Signs of the functions $N(a), D(a), d_6(a)$ when $0.1255 < \gamma^2 < 1/4$

- $\gamma^2 > \frac{1}{4}$

  In this case we conclude that a_1 is in region IV of Table 3, as is a_2, and a_3 is in region V. Thus again both $d(a)$ and $d_6(a)$ are positive in the region $a_1 < a < a_2$, and this is the case for all values of β^2. As noted previously, there is a further region $a > a_3$ in which positive-definitenesss is assured provided that $\beta^2 < N(a)/D(a)$. Since $N(a)$ grows with a more rapidly than $D(a)$, the upper bound on β^2 grows with a.

Region	a	$N(a)$	$D(a)$	$d_6(a)$
I	$a < a_4$	+	+	
II	$a_4 < a < a_-$	-	+	-
III	$a_- < a < a_5$	-	-	
IV	$a_5 < a < a_+$	+	-	+
V	$a_+ < a$	+	+	

Table 3: Signs of the functions $N(a), D(a), d_6(a)$ when $\gamma^2 > 1/4$. For $\gamma^2 < 1$ $a_5 = -1$, whereas for $\gamma^2 > 1$ $a_4 = -1$.

To solve the cubic we write $N(a)$, defined in equation (230) in the form

$$N(a) = (a+1)(4\gamma^2 - 1)C(a) \tag{295}$$

where $C(a)$ has the form

$$C(a) = a^3 + c_2 a^2 + c_1 a + c_0 \tag{296}$$

with

$$c_2 = \frac{1}{4\gamma^2 - 1} = -c_1 = c_0 \tag{297}$$

The roots of $C(a)$ are found by first changing variables from a to y, where

$$a = y - \frac{1}{3}c_2 \tag{298}$$

to cast it in the canonical form

$$y^3 + py = q \tag{299}$$

with

$$p \equiv c_1 - \frac{1}{3}c_2^2 = \frac{2(1-6\gamma^2)}{3(4\gamma^2-1)^2} \tag{300}$$

$$q \equiv \frac{1}{27}(9c_1c_2 - 27c_0 - 2c_2^3) = \frac{4(45\gamma^2 - 108\gamma^4 - 5)}{27(4\gamma^2-1)^3} \tag{301}$$

Equation (299) is solved by making Vieta's substitution

$$y = w - \frac{p}{3w} \tag{302}$$

so that

$$w^3 = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{4p^3}{27}} \tag{303}$$

$$= \frac{2(45\gamma^2 - 108\gamma^4 - 5)}{27(4\gamma^2-1)^3} \pm \frac{2}{(4\gamma^2-1)^2}\sqrt{\frac{27\gamma^4 - 10\gamma^2 + 1}{27}} \tag{304}$$

Thus we can solve for any value of $\gamma^2 \neq 1/4$. Consider first the plus sign in equation (304). For $0 \leq \gamma^2 < 1/4$, the root $\alpha(\gamma^2)$ of the cubic $C(a) = 0$ increases monotonically from $\alpha(0) = 1$ with $\alpha(\gamma^2) \to +\infty$ as $\gamma^2 \to 1/4$ from below. Thus in this case $a_5 > 1$. For $\gamma^2 > 1/4$, the root $\alpha(\gamma^2)$ is negative and monotonically increasing as γ^2 increases, with $\alpha(\gamma^2) \to -\infty$ as $\gamma^2 \to 1/4$ from above, and $\alpha(\gamma^2) \to 0$ as $\gamma^2 \to \infty$. For the solution corresponding to the minus sign in (304) one has to be more careful because $w(1/6) = 0 = p(1/6)$ which makes the evaluation of y undefined at that value of γ^2; one has to do the ranges $0 \leq \gamma^2 < 1/6$ and $1/6 < \gamma^2 < 1/4$ separately, even though there's nothing special about the cubic for this value of γ^2. However, the conclusion is that both signs give the same value of the root $\alpha(\gamma^2)$ for any given value of γ^2.

The following two examples illustrate the procedure.

- $\gamma^2 = 0.2$

  Since $\gamma^2 > 0.1955$, we expect positive-definiteness for some values of a in region IV of Table 2, besides those in $-0.7656 = a_1 < a < a_2 = 1$ in region III. For this value of γ^2 equations (226), (304), (302) and (298) yield

  $$a_3 = 3.266 \quad \text{and} \quad a_5 = 4.074 \tag{305}$$

  so for values a in the range $a_3 < a < a_5$, $d(a)$ is positive, and so are $N(a)$ and $D(a)$. Then it follows from (231) that $d_6(a)$ too is positive for values of β^2 satisfying

  $$\beta^2 < \frac{N(a)}{D(a)} = \frac{-a^4 + 4a^3 + 5}{7a^2 - 15} \tag{306}$$

  In this range the upper bound on β^2 is approximately linear, starting at $\beta^2 \lesssim 0.51$ when $a = a_3$ and decreasing to zero at $a = a_5$, where $N(a)$ vanishes.

- $\gamma^2 = 2$

  The solution of the cubic is not needed in this case. Besides the range $-0.3904 = a_1 < a < a_2 = 0.6404$ in region IV of Table 3, both $d_6(a)$ and $d(a)$ are positive in the range $a > a_3 = 1$ in region V, provided that $\beta^2 < \frac{7a^4+8a^3+1}{5a^2-3}$. Positive-definiteness is assured for all values of a in this range when $\beta^2 \lesssim 6.985$, whereas larger values of β^2 are only allowed for larger values of a.

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